

INTRODUCTION TO TRIGONOMETRY

1. Introduction

2. Trigonometric Ratios

3. Trigonometric Ratios of Some Specific Angles

4. Trigonometric Ratios of Complementary Angles

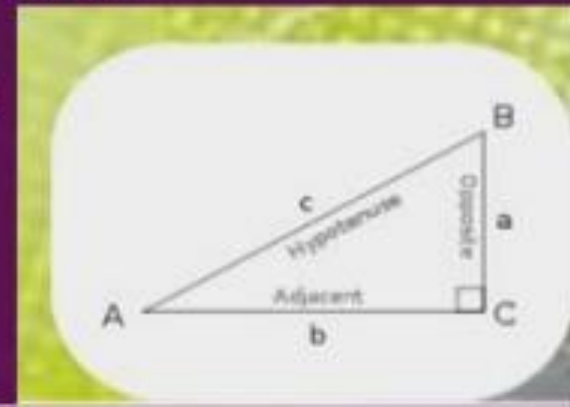
5. Trigonometric Identities

INTRODUCTION.....

- In this chapter, we will study some ratios of the sides of a right triangle with respect to its acute angles, called **TRIGNOMETRIC RATIOS OF THE ANGLE.**
- We will also define the trigonometric ratios for the angles of measures 0° and 90° . We will calculate trigonometric ratios for some specific angles and establish some identities involving these ratios, called **TRIGNOMETRIC IDENTITIES.**

What is trigonometry

- Trigonometry is a branch of mathematics that studies triangles and the relationship between their sides and the angles between these sides.
- The word 'TRIGONOMETRY' is derived from the Greek words 'TRI' (meaning three), 'GON' (meaning sides) and 'METRON' (meaning measure).

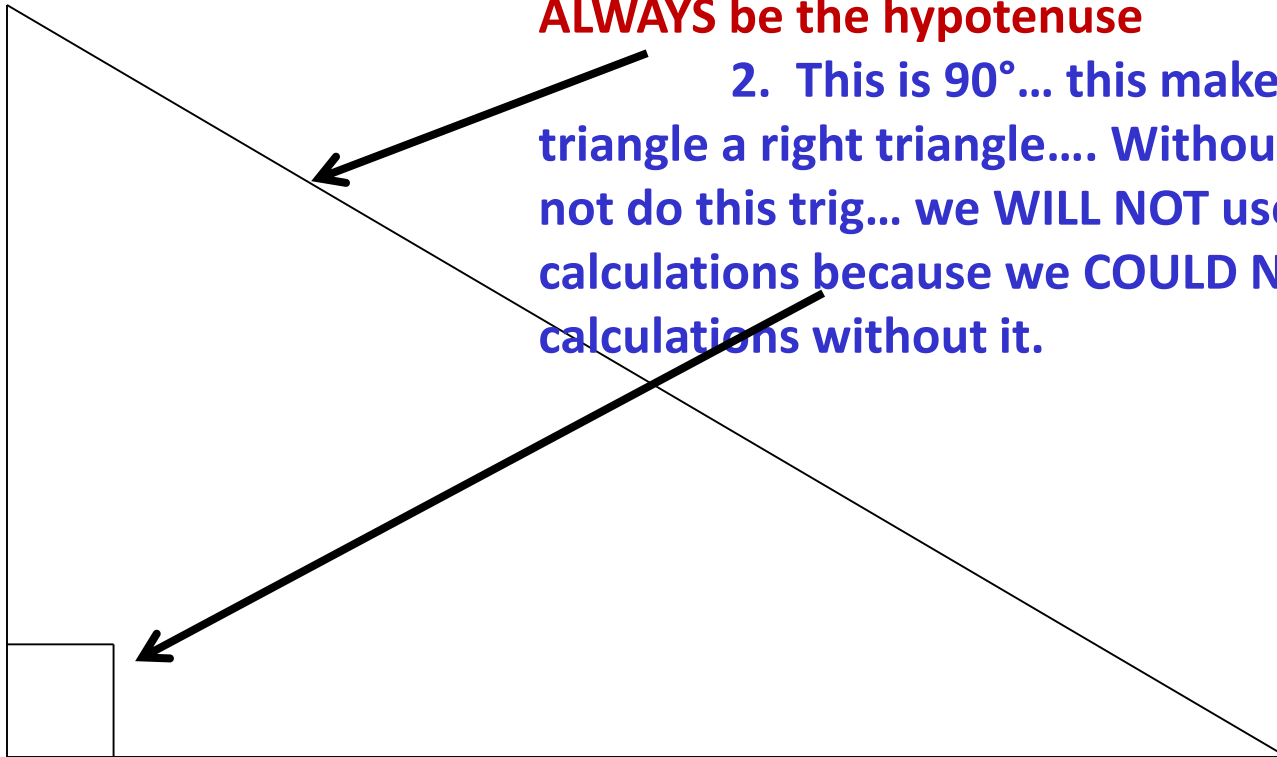


In trigonometry, the ratio we are talking about is the comparison of the sides of a **RIGHT TRIANGLE**.

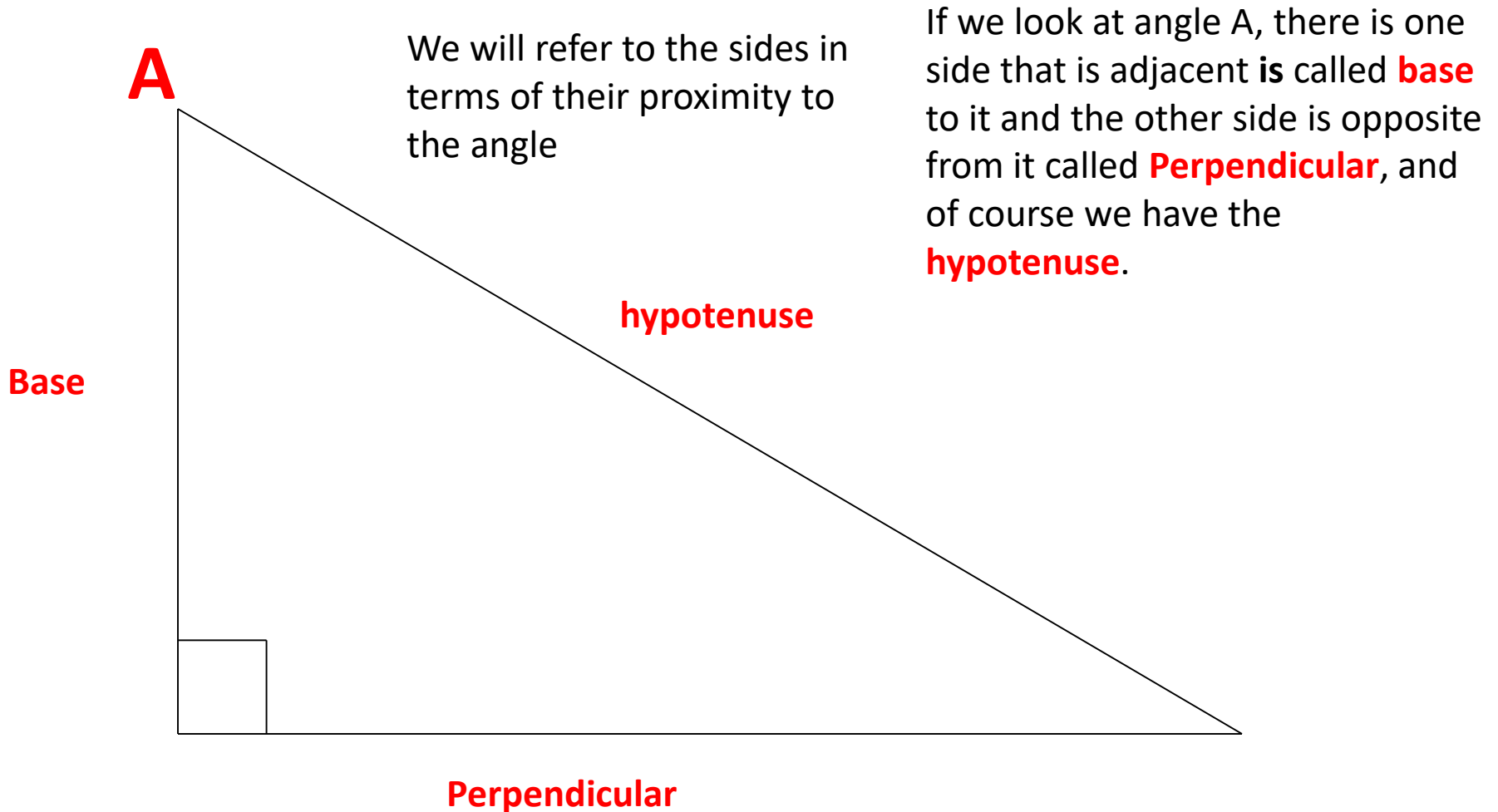
Two things MUST BE understood:

1. This is the hypotenuse.. This will ALWAYS be the hypotenuse

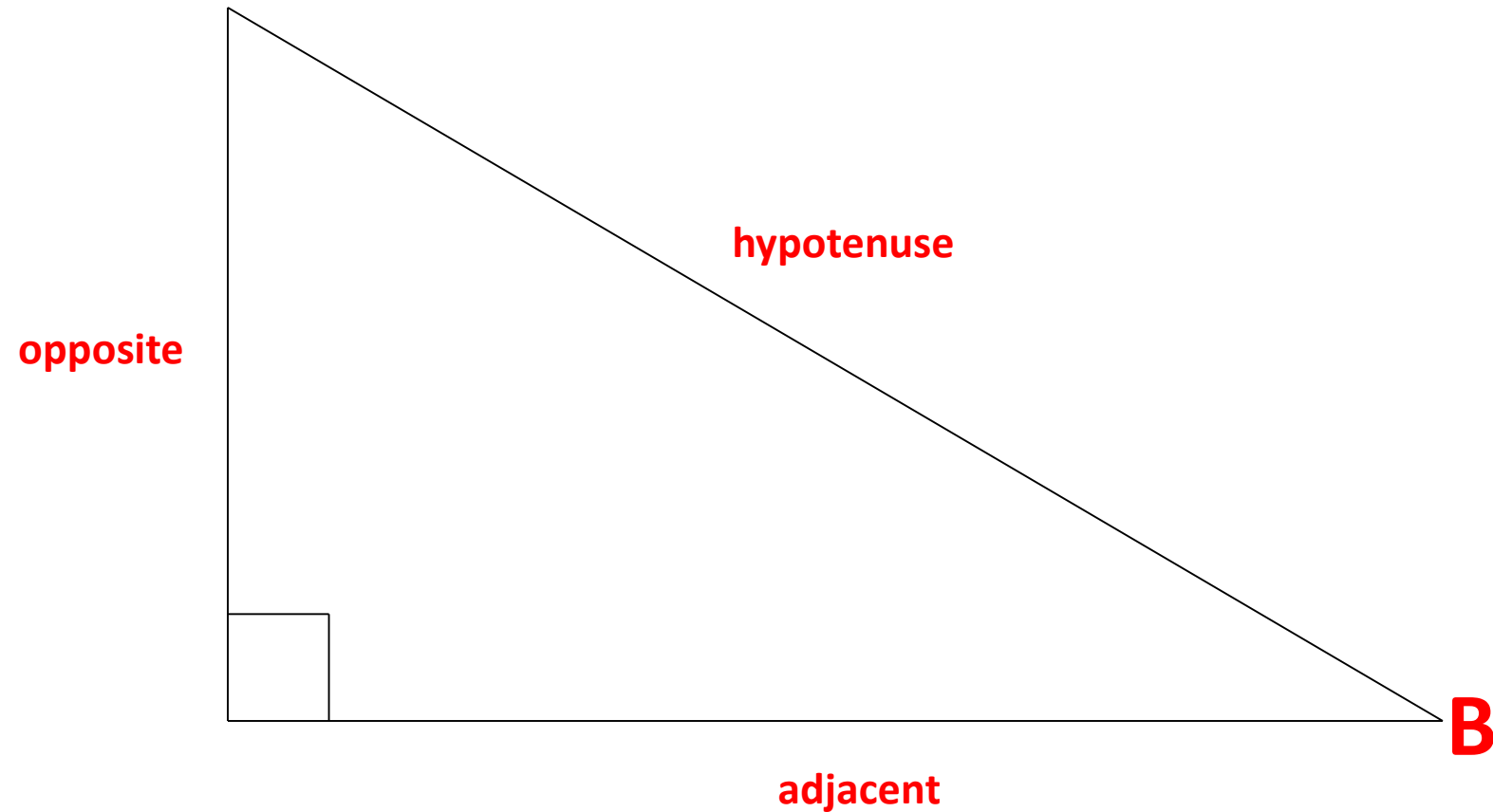
2. This is 90° ... this makes the right triangle a right triangle.... Without it, we can not do this trig... we WILL NOT use it in our calculations because we COULD NOT do calculations without it.



Now that we agree about the hypotenuse and right angle, there are only 4 things left; the 2 other angles and the 2 other sides.



If we look at angle B, there is one side that is **base** to it and the other side is **perpendicular** to it, and of course we have the **hypotenuse**.



TRIGONOMETRIC RATIOS

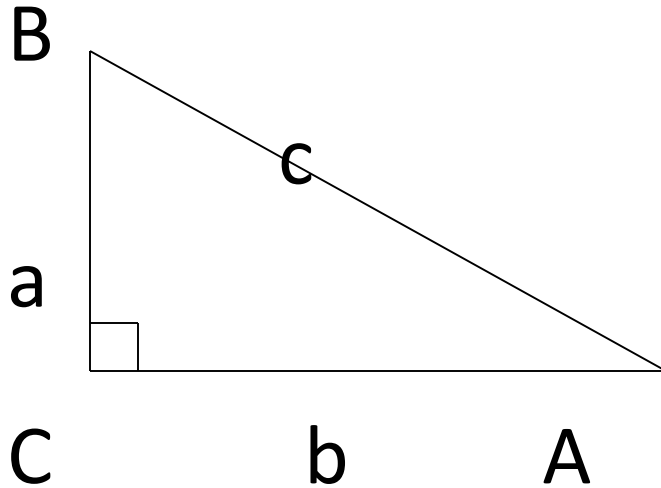
- Let us consider a right triangle:
- Here angle A is an acute angle. Note the position of the side BC with respect to angle A. we will call the side opposite to angle A. AC is the *hypotenuse of the right angled Δ* and the side AB is the part of angle A. so, we will call it as *the side adjacent to angle A*.
- The trigonometric ratios of angle A in right angle ABC are defined as follows :
- Sine of angle A = $\frac{\text{side opposite to angle A}}{\text{hypotenuse}} = \frac{BC}{AC}$
- Cosine of angle A = $\frac{\text{side adjacent to angle A}}{\text{hypotenuse}} = \frac{AB}{AC}$
- Tangent of angle A = $\frac{\text{side opposite to angle A}}{\text{side adjacent to angle A}} = \frac{BC}{AB}$

TRIGONOMETRIC RATIOS CONT.

- Cosecant of angle A = $\frac{\text{hypotenuse}}{\text{side opposite to angle A}} = \frac{AC}{BC}$
- Secant of angle A = $\frac{\text{hypotenuse}}{\text{side of adjacent to angle A}} = \frac{AC}{AB}$
- Cotangent of angle A = $\frac{\text{side adjacent to angle A}}{\text{side opposite to angle A}} = \frac{AB}{BC}$
- The ratios defined above are abbreviated as $\sin A$, $\cos A$, $\tan A$, $\operatorname{cosec} A$, $\sec A$, and $\cot A$ respectively. Note that the ratios **$\operatorname{cosec} A$, $\sec A$ and $\cot A$** are respectively, the reciprocals of the ratios $\sin A$, $\cos A$, and $\tan A$.
- Also, observe that $\tan A = \frac{BC/AC}{AB/AC} = \frac{\sin A}{\cos A}$ and $\cot A = \frac{\cos A}{\sin A}$

so, the trigonometric ratios of an acute angle in a right Δ express the relationship between the angle and the length of its sides.

Let's practice...



Write the ratio for sin A

$$\sin A = \frac{a}{c}$$

Write the ratio for cos A

$$\cos A = \frac{b}{c}$$

Write the ratio for tan A

$$\tan A = \frac{a}{b}$$

Let's switch angles:
Find the sin, cos and tan
for Angle B:

$$\sin B = \frac{b}{c}$$

$$\cos B = \frac{a}{c}$$

$$\tan B = \frac{b}{a}$$

- **LET US TRY AN EXAMPLE :**
- QUESTION : given $\tan A = 4/3$, find the other trigonometric ratios of the angle A.
- SOLUTION : let us first draw a right ΔABC

$$\text{now we know that } \tan A = \frac{BC}{AB} = \frac{4}{3} = \frac{\text{opp.}}{\text{adj.}}$$

therefore, to find hypotenuse we use Pythagoras theorem,
 $AC^2 = AB^2 + BC^2 = (3)^2 + (4)^2 = 9 + 16 = 25$
 $AC = \sqrt{25} = 5$ cm

now, we can write all the trigonometric ratios using their definitions.

$$\sin A = \frac{BC}{AC} = \frac{4}{5}$$

$$\cos A = \frac{AB}{AC} = \frac{3}{5}$$

Therefore, $\cot A = \frac{3}{4}$, $\operatorname{cosec} A = \frac{5}{4}$, and $\sec A = \frac{5}{3}$

TRIGONOMETRIC RATIOS OF SOME SPECIFIC ANGLES

- Trigonometric ratios of 45°**

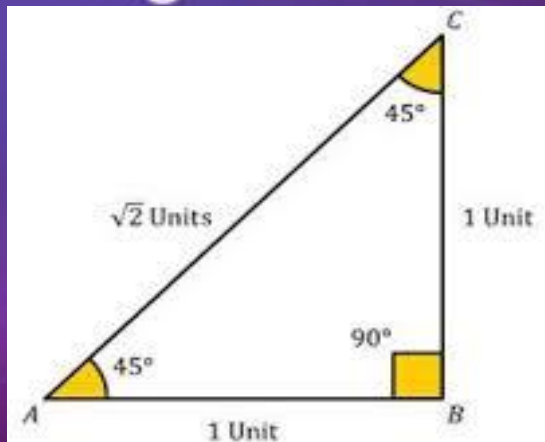
- In ΔABC , right-angled at B, if one angle is 45° , then other angle is also 45°

- So, $BC = AB$

- Now, suppose $BC = AB = a$.

- Then, by Pythagoras theorem, $AC^2 = BC^2 + AB^2 = a^2 + a^2 = 2a^2$

- Using definitions : $\sin 45^\circ = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$



$$\sin 45^\circ = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

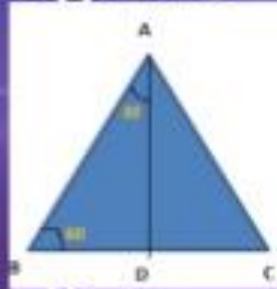
$$\tan 45^\circ = \frac{BC}{AB} = \frac{a}{a} = 1$$

$$\cot 45^\circ = \frac{AB}{BC} = \frac{a}{a} = 1$$

$$\sec 45^\circ = \frac{AC}{AB} = \frac{a\sqrt{2}}{a} = \sqrt{2}$$

- Also, $\operatorname{cosec} 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2}$, $\sec 45^\circ = \sqrt{2}$, $\cot 45^\circ = \frac{1}{\tan 45^\circ} = 1$

- **Trigonometric ratios of 30° and 60°**
- Consider an equilateral $\triangle ABC$. Since each angle is equal to 60° .
- So, $AB = BC = AC$.
- Now, $\triangle ABC \cong \triangle ACD$ (AD is a median and median divides a $\triangle$ into two equal parts).
- Therefore, $BD = DC$ and $\angle BAD = \angle CAD$ (CPCT)
- Now, $\triangle ABD$ is a right angled triangle with $\angle BAD = 30^\circ$ and $\angle ABD = 60^\circ$
- As we know, for finding trigonometric ratios, we need to know the lengths of the sides of the $\triangle$. So, let us suppose that $AB = 2a$.



- **Trigonometric ratios of 30° and 60° cont.....**

- Then, $BD = \frac{1}{2} BC = a$

And $AD^2 = AB^2 - BD^2 = (2a)^2 - (a)^2 = 3a^2,$

Now we have : $\sin 30^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}, \cos 30^\circ = \frac{AD}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2},$
 $\tan 30^\circ = \frac{BD}{AD} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\frac{AD}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$

Also, $\operatorname{cosec} 30^\circ = \frac{1}{\sin 30^\circ} = 2, \sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}}, \cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}$

$\sin 30^\circ$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$

Similarly, $\sin 60^\circ = \frac{AD}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}, \cos 60^\circ = \frac{1}{2}, \tan 60^\circ = \sqrt{3},$
 $\operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}},$

$\frac{AB}{AD} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}$

$\frac{1}{2}$

$\frac{\sqrt{3}}{2}$

$\sec 60^\circ = 2$ and $\cot 60^\circ = \frac{1}{\sqrt{3}}$.

- Trigonometric ratios of 0° and 90°
- Let us see what happens to the trigonometric ratios of angle A, if it is made smaller and smaller in the right triangle ABC, till it becomes zero. As $\angle A$ gets smaller and smaller, the length of the side BC decreases. The point C gets closer to point B, and finally when $\angle A$ becomes very close to 0° , AC becomes almost the same as AB.

Trigonometric ratios of 0° and 90° cont..

- When $\angle A$ is very close to 0° , BC gets very close to 0 and so the value of $\sin A = BC/AC$ is very close to 0. Also, when $\angle A$ is very close to 0° , AC is nearly the same as AB and so the value of $\cos A = AB/AC$ is very close to 1.
- This helps us to see how we can define the values of $\sin A$ and $\cos A$ when $A = 0^\circ$. We define : $\sin 0^\circ = 0$ and $\cos 0^\circ = 1$.
- $\sin 0^\circ = 0$ and $\cos 0^\circ = 1$.
- $\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = 0$, $\cot 0^\circ = \frac{1}{\tan 0^\circ}$, which is not defined.

- **Trigonometric ratios of 0° and 90° cont.....**
- $\sec 0^\circ = \frac{1}{\cos 0^\circ} = 1$ and $\operatorname{cosec} 0^\circ = \frac{1}{\sin 0^\circ}$, which is again not defined.
- Now, let us see what happens to the trigonometric ratios of $\angle A$, when it is made larger and larger in ΔABC till it becomes 90° . As $\angle A$ gets larger and larger, $\angle C$ gets smaller and smaller. Therefore, as in the case above, the length of the side AB goes on decreasing. The point A gets closer to point B. Finally when $\angle A$ is very close to 90° , $\angle C$ becomes very close to 0° and the side AC almost coincides with side BC.

Trigonometric ratios of 0° and 90° cont.....

- When $\angle C$ is very close to 0° , $\angle A$ is very close to 90° , side AC is nearly the same as side BC, and so $\sin A$ is very close to 1. Also when $\angle A$ is very close to 90° ,
- $\angle C$ is very close to 0° , and the side AB is nearly zero, so $\cos A$ is very close to 0.
- So, we define : **$\sin 90^\circ = 1$ and $\cos 90^\circ = 0$**

TRIGONOMETRIC RATIOS OF SOME SPECIFIC ANGLES TABLE.....

θ	0°	30°	45°	60°	90°
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\cot \theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec \theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\operatorname{cosec} \theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Trigonometric ratios of complementary angles

- As we know that in $\triangle ABC$, right-angled at B, we can see 2 complementary angles.
- Since $\angle A + \angle C = 90^\circ$, they form such a pair. We have —
- $\sin A = \frac{BC}{AC}$, $\cos A = \frac{AB}{AC}$, $\tan A = \frac{BC}{AB}$,
- $\operatorname{cosec} A = \frac{AC}{BC}$, $\sec A = \frac{AC}{AB}$, $\cot A = \frac{AB}{BC}$ — ①

Trigonometric ratios of complementary angles cont.

Now let us write the trigonometric ratios for $\angle C = 90^\circ - \angle A$. For convenience, we shall write $90^\circ - A$ instead of $90^\circ - \angle A$. What would be the side opposite and the side adjacent to the angle $90^\circ - A$?

You will find that AB is the side opposite and BC is the side adjacent to the angle

$90^\circ - A$. Therefore,

$$\sin (90^\circ - A) = \frac{AB}{AC}, \cos (90^\circ - A) = \frac{BC}{AC}, \tan (90^\circ - A) = \frac{AB}{BC}$$

$$\operatorname{Cosec} (90^\circ - A) = \frac{AC}{AB}, \sec (90^\circ - A) = \frac{AC}{BC}, \cot (90^\circ - A) = \frac{BC}{AB} \quad \text{--- (2)}$$

- Now, compare the ratios in (1) and (2). Observe that :
- $\sin (90^\circ - A) = \frac{AB}{AC} = \cos A$ and $\cos (90^\circ - A) = \frac{BC}{AC} = \sin A$

Trigonometric ratios of complementary angles cont.

- Also, $\tan (90^\circ - A) = \frac{AB}{BC} = \cot A$, $\cot (90^\circ - A) = \frac{BC}{AB} = \tan A$
- $\sec (90^\circ - A) = \frac{AC}{BC} = \operatorname{cosec} A$, $\operatorname{cosec} (90^\circ - A) = \frac{AC}{AB} = \sec A$
- So, $\sin (90^\circ - A) = \cos A$, $\cos (90^\circ - A) = \sin A$,
- $\tan (90^\circ - A) = \cot A$, $\cot (90^\circ - A) = \tan A$,
- $\sec (90^\circ - A) = \operatorname{cosec} A$, $\operatorname{cosec} (90^\circ - A) = \sec A$,
- for all values of angle A lying between 0° and 90° . Check whether this holds for $A = 0^\circ$ or $A = 90^\circ$.
- Note : $\tan 0^\circ = 0 = \cot 90^\circ$, $\sec 0^\circ = 1 = \operatorname{cosec} 90^\circ$ and $\sec 90^\circ$, $\operatorname{cosec} 0^\circ$, $\tan 90^\circ$ and $\cot 0^\circ$ are not defined.

TRIGONOMETRIC RATIOS

- In ΔABC , right-angled at B, we have: $AB^2 + BC^2 = AC^2$
(1)

Dividing each term of (1) by AC^2 , we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

$$\text{i.e., } (AB/AC)^2 + (BC/AC)^2 = (AC^2/AC^2)$$

$$\text{i.e., } (\cos A)^2 + (\sin A)^2 = 1$$

$$\text{i.e., } \cos^2 A + \sin^2 A = 1 \quad (2)$$

This is true for all A such that $0^\circ \leq A \leq 90^\circ$. So, this is a trigonometric identity.

Let us now divide (1) by AB^2

Cont.....

- $\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$

$$\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$

or, $(AB/AB)^2 + (BC/AB)^2 = (AC/AB)^2$

i.e., **$1 + \tan^2 A = \sec^2 A$** (3)

Is this equation true for $A = 0^\circ$? Yes, it is. What about $A = 90^\circ$?

Well, $\tan A$ and $\sec A$ are not defined for $A = 90^\circ$. So, (3) is true for all A such that $0^\circ \leq A < 90^\circ$. Let us see what we get on dividing (1) by BC^2 . We get

$$\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$$

$$\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$$

i.e., $(AB^2/BC^2) + (BC^2/BC^2) = (AC^2/BC^2)$

i.e., **$\cot^2 A + 1 = \operatorname{cosec}^2 A$** (4)

Cont.....

Note that cosec A and cot A are not defined for $A = 0^\circ$. Therefore (4) is true for all A such that $0^\circ < A \leq 90^\circ$.

Using these identities, we can express each trigonometric ratio in terms of other trigonometric ratios, i.e., if any one of the ratios is known, we can also determine the values of other trigonometric ratios.

Let us see how we can do this using these identities. Suppose we know that

- $\tan A = \frac{1}{\sqrt{3}}$, then $\cot A = \sqrt{3}$

Since, $\sec^2 A = 1 + \tan^2 A = 1 + \frac{1}{3} = \frac{4}{3}$, $\sec A = \frac{2}{\sqrt{3}}$ and $\cos A = \frac{\sqrt{3}}{2}$

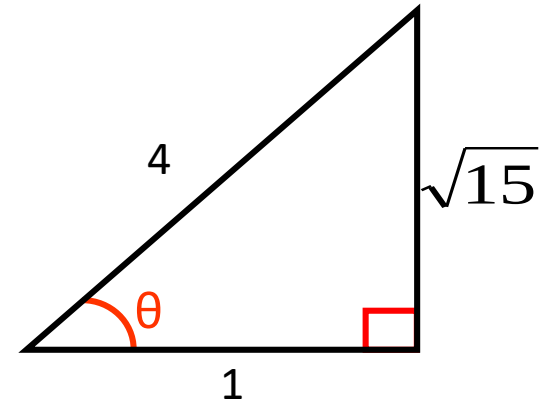
Again $\sin A = \sqrt{1 - \cos^2 A} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$ Therefore, cosec A = 2.

Example: Given $\sec \theta = 4$, find the values of the other five trigonometric functions of θ .

Solution:

Draw a right triangle with an angle θ such

$$\text{that } 4 = \sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{4}{1}.$$



Use the Pythagorean Theorem to solve for the third side of the triangle.

$$\sin \theta = \frac{\sqrt{15}}{4}$$

$$\cos \theta = \frac{1}{4}$$

$$\tan \theta = \frac{\sqrt{15}}{1} = \sqrt{15}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{4}{\sqrt{15}}$$

$$\sec \theta = \frac{1}{\cos \theta} = 4$$

$$\cot \theta = \frac{1}{\sqrt{15}}$$