

Relations And Functions

A **relation** is a set of ordered

The **domain** is the set of all x values in the relation

$$\text{domain} = \{-1, 0, 2, 4, 9\}$$

These are the x values written in a set from smallest to largest

$$\{(2, 3), (-1, 5), (4, -2), (9, 9), (0, -6)\}$$

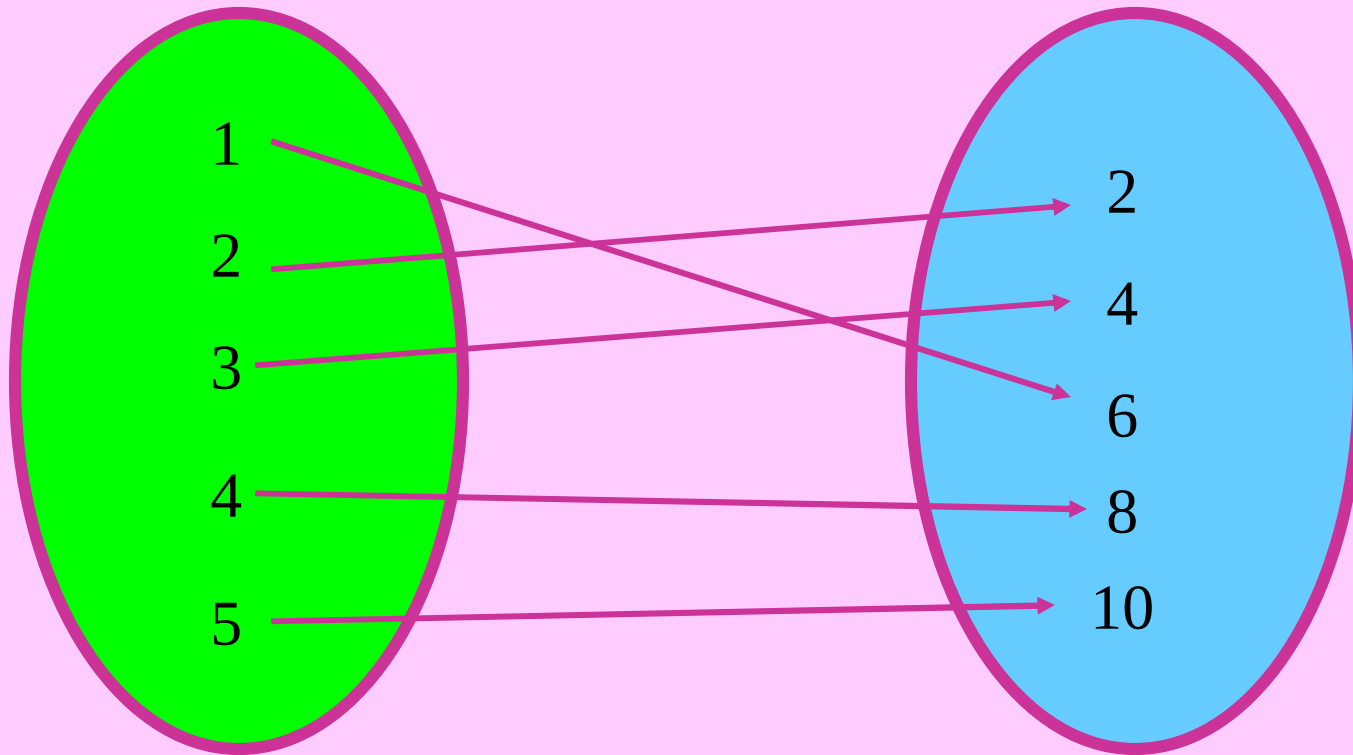
← This is a relation

These are the y values written in a set from smallest to largest

$$\text{range} = \{-6, -2, 3, 5, 9\}$$

The **range** is the set of all y values in the relation

A relation assigns the x's with y's

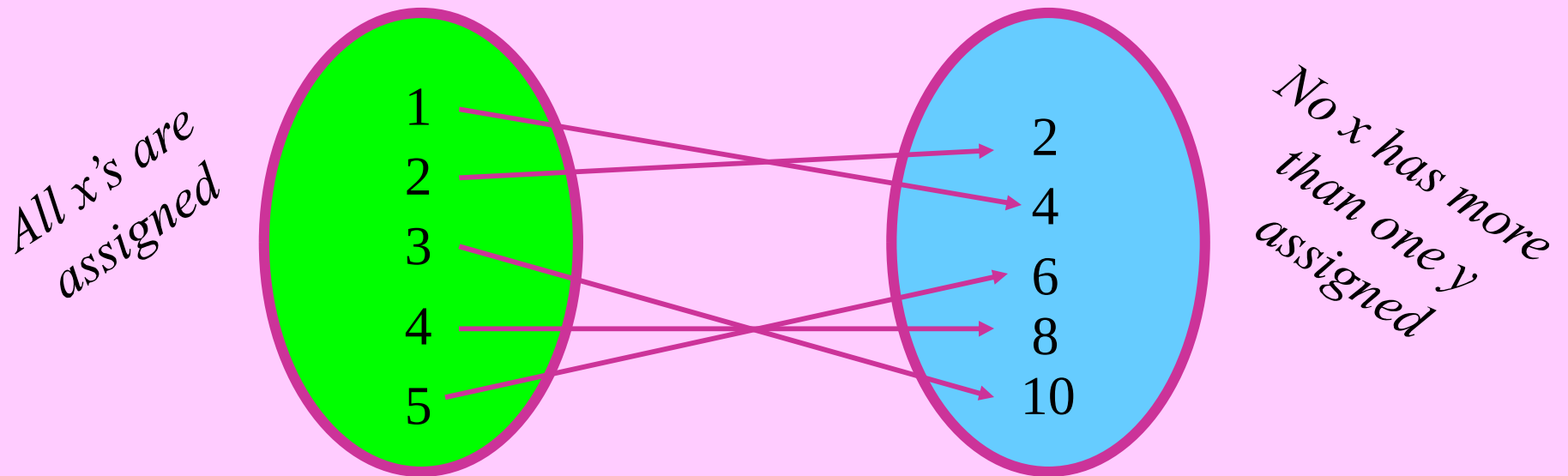


Domain (set of all x's)

Range (set of all y's)

This relation can be written $\{(1,6), (2,2), (3,4), (4,8), (5,10)\}$

A function f from set A to set B is a rule of correspondence that assigns to each element x in the set A **exactly one element y** in the set B.



Set A is the domain

Set B is the range

*This is a function
---it meets our
conditions*

Must use all the x 's

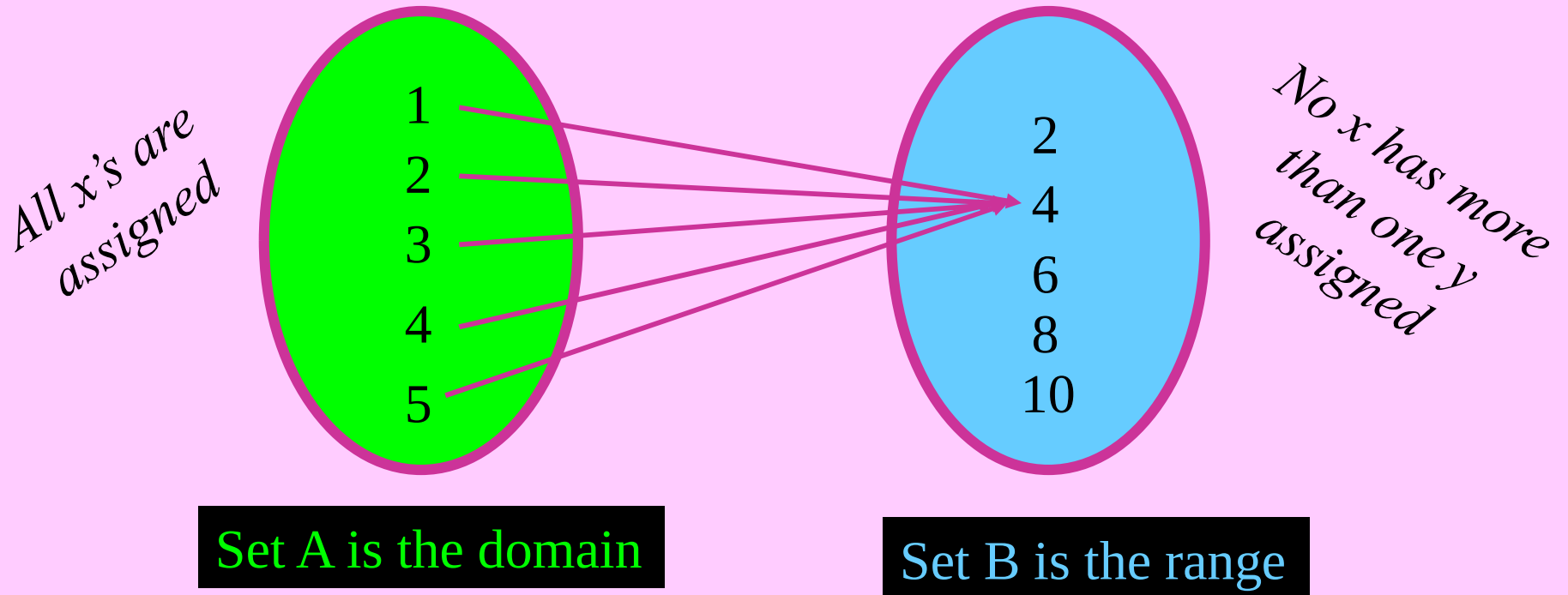


The x value can only be assigned to one y



Let's look at another relation and decide if it is a function.

The second condition says each x can have only one y , but it CAN be the same y as another x gets assigned to.



This is a function
---it meets our
conditions

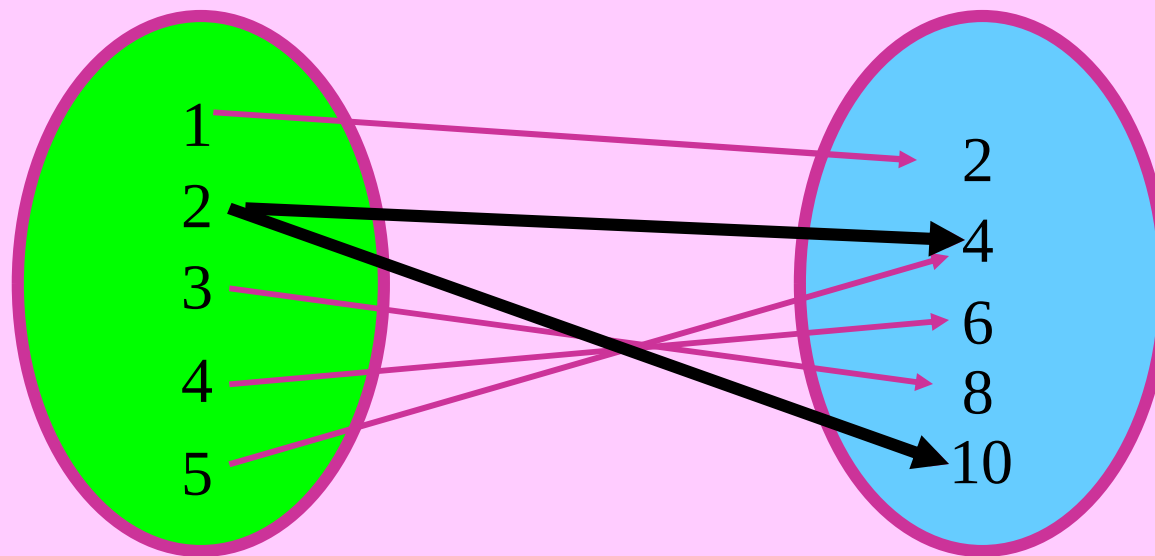
Must use all the x 's



The x value can only be assigned to one y



A good example that you can “relate” to is students in our maths class this semester are set A. The grade they earn out of the class is set B. Each student must be assigned a grade and **can only be assigned ONE grade**, but more than one student can get the same grade (we hope so---we want lots of A’s). The example shown on the previous screen had each student getting the same grade. That’s okay.



2 was assigned both 4 and 10

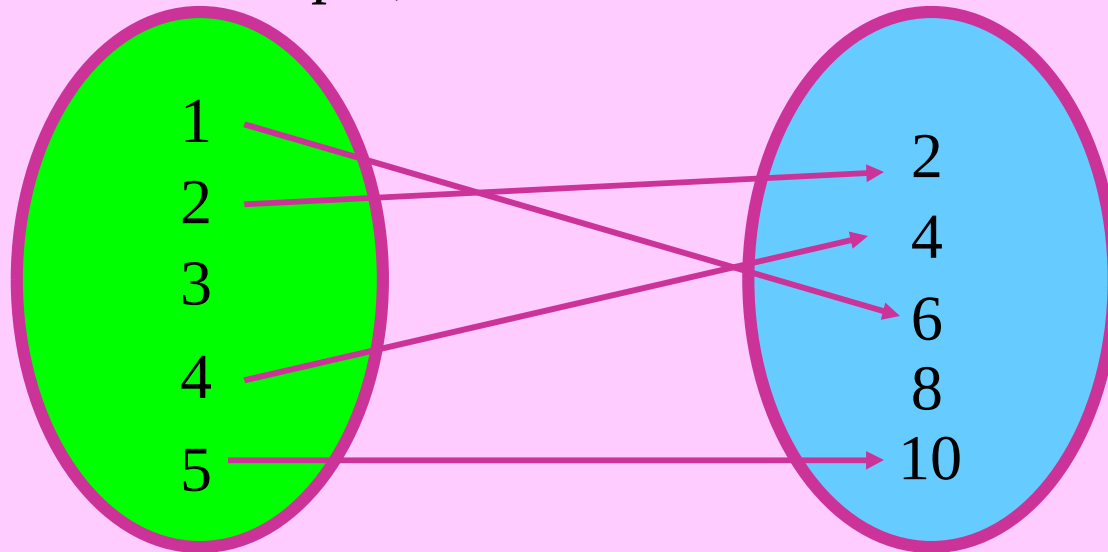
Is the relation shown above a function? **NO**

Why not???

Check this relation out to determine if it is a function.

It is not---3 didn't get assigned to anything

Comparing to our example, a student in maths must receive a grade



Set A is the domain

Set B is the range

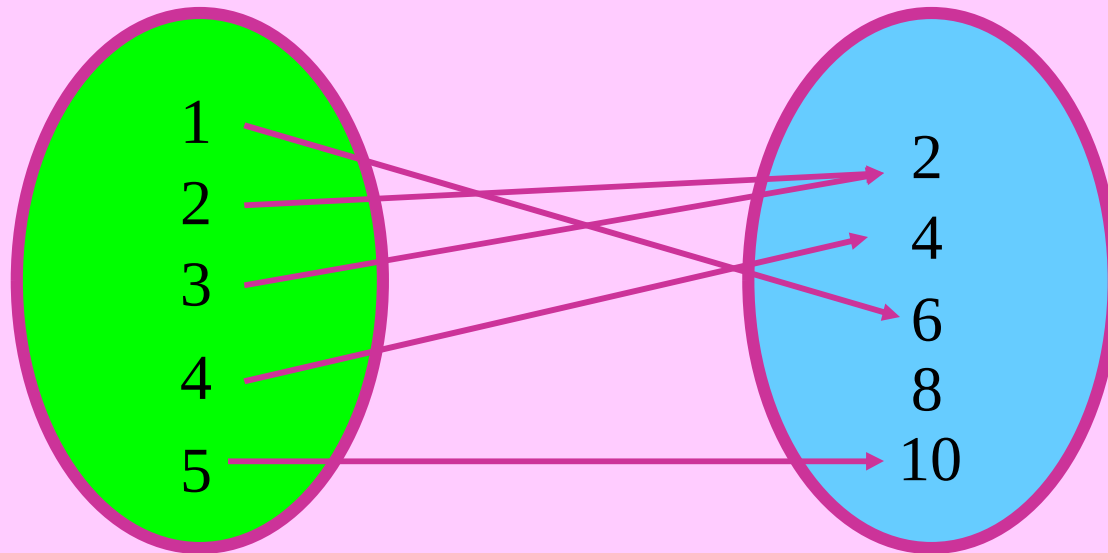
This is not a function---it doesn't assign each x with a y

Must use all the x 's

The x value can only be assigned to one y

Check this relation out to determine if it is a function.

This is fine—each student gets only one grade. More than one can get an A and I don't have to give any D's (so all y 's don't need to be used).



Set A is the domain

Set B is the range

This is a function

Must use all the x 's

The x value can only be assigned to one y

Function Notation

We commonly call functions by letters. Because function starts with f , it is a commonly used letter to refer to functions.

$$f(x) = 2x^2 - 3x + 6$$

This means
the right
hand side is
a function
called f

This means
the right hand
side has the
variable x in it

*The left side DOES NOT
MEAN f times x like
brackets usually do, it
simply tells us what is on
the right hand side.*

The left hand side of this equation is the function notation. It tells us two things. We called the function f and the variable in the function is x .

Remember---this tells you what is on the right hand side---it is not something you work. It says that the right hand side is the function f and it has x in it.

$$f(x) = 2x^2 - 3x + 6$$
$$f(2) = 2(2)^2 - 3(2) + 6$$

$$f(2) = 2(4) - 3(2) + 6 = 8 - 6 + 6 = 8$$

So we have a function called f that has the variable x in it.

Using function notation we could then ask the following:

Find $f(2)$.

This means to find the function f and instead of having an x in it, put a 2 in it. So let's take the function above and make brackets everywhere the x was and in its place, put in a 2.

Don't forget order of operations---powers, then multiplication, finally addition & subtraction

Find $f(-2)$.

$$f(x) = 2x^2 - 3x + 6$$

$$f(-2) = 2(-2)^2 - 3(-2) + 6$$

$$f(-2) = 2(4) - 3(-2) + 6 = 8 + 6 + 6 = 20$$

This means to find the function f and instead of having an x in it, put a -2 in it. So let's take the function above and make brackets everywhere the x was and in its place, put in a -2 .

Don't forget order of operations---powers, then multiplication, finally addition & subtraction

Find $f(k)$.

$$f(x) = 2x^2 - 3x + 6$$

$$f(k) = 2(k)^2 - 3(k) + 6$$

$$f(k) = 2(k^2) - 3(k) + 6 = 2k^2 - 3k + 6$$

This means to find the function f and instead of having an x in it, put a k in it. So let's take the function above and make brackets everywhere the x was and in its place, put in a k .

Don't forget order of operations---powers, then multiplication, finally addition & subtraction

Find $f(2k)$.

$$f(x) = 2x^2 - 3x + 6$$

$$f(2k) = 2(2k)^2 - 3(2k) + 6$$

$$f(2k) = 2(4k^2) - 3(2k) + 6 = 8k^2 - 6k + 6$$

This means to find the function f and instead of having an x in it, put a $2k$ in it. So let's take the function above and make brackets everywhere the x was and in its place, put in a $2k$.

Don't forget order of operations---powers, then multiplication, finally addition & subtraction

Let's try a new function

Find $g(1) + g(-4)$.

$$g(x) = x^2 - 2x$$

$$g(1) = (1)^2 - 2(1) = -1$$

$$g(-4) = (-4)^2 - 2(-4) = 16 + 8 = 24$$

$$\text{So } g(1) + g(-4) = -1 + 24 = 23$$

The last thing we need to learn about functions for this section is something about their domain. Recall domain meant "Set A" which is the set of values you plug in for x .

For the functions we will be dealing with, there are two "illegals":

1. You can't divide by zero (denominator (bottom) of a fraction can't be zero)
2. You can't take the square root (or even root) of a negative number

When you are asked to find the domain of a function, you can use any value for x as long as the value won't create an "illegal" situation.

Find the domain for the following functions:

$$f(x) = 2x - 1$$

Note: There is nothing wrong with the top = 0 just means the fraction = 0

$$g(x) = \frac{x + 3}{x - 2}$$

illegal if this is zero

Since no matter what value you choose for x , you won't be dividing by zero or square rooting a negative number, you can use anything you want so we say the answer is:

All real numbers x .

If you choose $x = 2$, the denominator will be $2 - 2 = 0$ which is illegal because you can't divide by zero. The answer then is:

All real numbers x such that $x \neq 2$.

means does not equal

Let's find the domain of another one:

$$h(x) = \sqrt{x - 4}$$

Can't be negative so must be ≥ 0

$$x - 4 \geq 0 \xleftarrow{\text{solve this}} \xrightarrow{\quad} x \geq 4$$

We have to be careful what x 's we use so that the second "illegal" of square rooting a negative doesn't happen. This means the "stuff" under the square root must be greater than or equal to zero (maths way of saying "not negative").

So the answer is:

All real numbers x such that $x \geq 4$

Summary of How to Find the Domain of a Function

- Look for any fractions or square roots that could cause one of the two "illegals" to happen. If there aren't any, then the domain is **All real numbers x** .
- If there are fractions, figure out what values would make the bottom equal zero and those are the values you can't use. The answer would be: **All real numbers x such that $x \neq$ those values**.
- If there is a square root, the "stuff" under the square root cannot be negative so set the stuff ≥ 0 and solve. Then answer would be: **All real numbers x such that $x \neq$ whatever you got when you solved**.

NOTE: Of course your variable doesn't have to be x , can be whatever is in the problem.

Class Exercise -

If $A = \{1, 2, 3\}$ and $B = \{3, 8\}$, then

$(A \cup B) \times (A \cap B)$ is

(a) $\{(3, 1), (3, 2), (3, 3), (3, 8)\}$

(b) $\{(1, 3), (2, 3), (3, 3), (8, 3)\}$

(c) $\{(1, 2), (2, 2), (3, 3), (8, 8)\}$

(d) $\{(8, 3), (8, 2), (8, 1), (8, 8)\}$

Solution

$$(A \cup B) = \{1, 2, 3, 8\}$$

$$A \cap B = \{3\}$$

$$(A \cup B) \times (A \cap B) = \{(1, 3), (2, 3), (3, 3), (8, 3)\}$$

Hence, answer is (b).

Class Exercise -

Let A and B be two non-empty sets

such that $|A \cap B| = n$

Then prove that $A \times B$ and $B \times A$ have n^2 elements in common.

Solution

$$\text{As } (A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

$$\begin{aligned}\therefore (A \times B) \cap (B \times A) &= (A \cap B) \times (B \cap A) \\ &= (A \cap B) \times (A \cap B)\end{aligned}$$

$$\begin{aligned}\therefore O((A \times B) \cap (B \times A)) &= O((A \cap B) \times (A \cap B)) \\ &= O(A \cap B) \cdot O(A \cap B) \\ &= (O(A \cap B))^2 = n^2\end{aligned}$$

$\Rightarrow A \times B$ and $B \times A$ have n^2 elements is common.

Class Exercise -

R be a relation on set of natural numbers N defined as $R = \{(a, b) \mid 3a + b = 12, a, b \in N\}$

Find the following.

- (i) R, R^{-1} as sets of ordered pairs
- (ii) Domain of R and R^{-1}
- (iii) Range of R and R^{-1}
- (iv) $R^{-1} \circ R$ (v) R^{-1} in set-builder form

Solution

$$(i) \ R = \{(a, b) \mid 3a + b = 12, a, b \in \mathbb{N}\} \\ = \{(1, 9), (2, 6), (3, 3)\}$$

$$R^{-1} = \{(9, 1), (6, 2), (3, 3)\}$$

$$(ii) \ \text{Domain}(R) = \{1, 2, 3\} \\ \text{Domain}(R^{-1}) = \{9, 6, 3\}$$

$$(iii) \ \text{Range}(R) = \{9, 6, 3\} \\ \text{Domain}(R^{-1}) = \{1, 2, 3\}$$

$$(iv) \ \{(9, 1), (6, 2), (3, 3)\} \circ \{(1, 9), (2, 6), (3, 3)\} \\ = \{(1, 1), (2, 2), (3, 3)\}$$

$$(v) \ R^{-1} = \{(a, b) \mid a + 3b = 12, a, \} \quad b \in \mathbb{N}$$

Class Exercise -

Let R be a relation from $A = \{2, 3, 4, 5, 6\}$ to $B = \{3, 6, 8, 9, 12\}$ defined as

$$R = \{(x, y) \mid x \in A, y \in B, x \mid y\}.$$

Express R as set of ordered pairs, find the domain and the range of R , and also find R^{-1} in set-builder form (where $x \mid y$ means x divides y).

Solution

$$\begin{aligned} R &= \{(x, y) \mid x \in A, y \in B, x \mid y\} \\ &= \{(2, 6), (2, 8), (2, 12), (3, 3), (3, 6), (3, 9), (3, 12), \\ &\quad (4, 8), (4, 12), (6, 6), (6, 12)\} \end{aligned}$$

$$\text{Domain}(R) = \{2, 3, 4, 6\}$$

$$\text{Range}(R) = \{6, 8, 12, 3, 9\}$$

$$\begin{aligned} R^{-1} &= \{(6, 2), (8, 2), (12, 2), (3, 3), (6, 3), (9, 3), \\ &\quad (12, 3), (8, 4), (12, 4), (6, 6), (12, 6)\} \\ &= \{(x, y) \mid x \in B, y \in A, \end{aligned}$$

and R^{-1} x is divisible by y

Class Exercise -

Let R be a relation on Z defined as $R = \{(x, y) \mid x, y \in Z, x^2 + y^2 = 25\}$.

Express R and R^{-1} as set of ordered pairs.

Hence, find the domain of R and R^{-1} .

Solution

$$R = \{(0, 5), (0, -5), (3, 4), (3, -4), (4, 3), (4, -3), \\ (5, 0), (-3, 4), (-3, -4), (-4, 3), (-4, -3), (-5, 0)\}$$

$$R^{-1} = \{(5, 0), (-5, 0), (4, 3), (-4, 3), (3, 4), (-3, 4), \\ (0, 5), (4, -3), (-4, -3), (3, -4), (-3, -4), (0, -5)\} = R$$

$$\therefore \text{Domain}(R) = \text{Domain}(R^{-1}) = \{0, 3, 4, 5, -3, -4, -5\}$$

Class Exercise -

Let S be the set of all the straight lines on a plane, R be a relation on S defined as

$$R = \{(l_1, l_2) \mid l_1 \perp l_2, l_1, l_2 \in S\}$$

Then check R for reflexivity, symmetry and transitivity.

Solution

Reflexive: $(l_1, l_1) \notin R$ for any $l_1 \in S$

as a line cannot be perpendicular to itself.

Symmetric: Let $(l_1, l_2) \in R$ i.e. $l_1 \perp l_2$

$$\Rightarrow l_2 \perp l_1$$

$$\Rightarrow (l_2, l_1) \in R$$

$\Rightarrow R$ is symmetric

Transitive: Let $(l_1, l_2), (l_2, l_3) \in R$ $l_1 \perp l_2$ and $l_2 \perp l_3$

$$\Rightarrow (l_1, l_3) \notin R \text{ as } l_1 \parallel l_3.$$

$\Rightarrow$ Not transitive

Class Exercise -

Let A be a set of all the points in space. Let R be a relation on A such that $a_1 R a_2$ if distance between the points a_1 and a_2 is less than one unit. Then which of the following is false?

- (a) R is reflexive*
- (b) R is symmetric*
- (c) R is transitive*
- (d) R is not an equivalence relation*

Class Exercise - 10

Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by

$$(a, b)R(c, d) \Rightarrow ad(b + c) = bc(a + d).$$

Show that R is an equivalence relation.

Solution

Reflexive: $(a, b) R (a, b)$

$$\text{As } ab(b + a) = ba(a + b)$$

$\Rightarrow$ Reflexive

Symmetric: If $(a, b) R (c, d)$

$$\Rightarrow ad(b + c) = bc(a + d)$$

$$\Rightarrow cb(d + a) = da(c + b)$$

$$\Rightarrow (c, d) R (a, b)$$

$\Rightarrow R$ is symmetric

Solution contd..

Transitive: If $(a, b) R (c, d)$ and $(c, d) R (e, f)$

$$\Rightarrow ad(b + c) = bc(a + d) \text{ and } cf(d + e) = de(c + f)$$

$$\Rightarrow \frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{d} \text{ and } \frac{1}{d} + \frac{1}{e} = \frac{1}{c} + \frac{1}{f}$$

$$\Rightarrow \frac{1}{c} - \frac{1}{d} = \frac{1}{a} - \frac{1}{b} \text{ and } \frac{1}{c} - \frac{1}{d} = \frac{1}{e} - \frac{1}{f} \Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{e} - \frac{1}{f} \Rightarrow \frac{1}{a} + \frac{1}{f} = \frac{1}{b} + \frac{1}{e}$$

$$\Rightarrow af(b + e) = be(a + f) \Rightarrow (a, b) R (e, f)$$

$\Rightarrow$ Transitive

Hence, R is an equivalence relation.