

Matrices



Matrix - a rectangular array of variables or constants in horizontal rows and vertical columns enclosed in brackets.

Element - each value in a matrix; either a number or a constant.

Dimension - number of rows by number of columns of a matrix.

****A matrix is named by its dimensions.**

Examples: Find the dimensions of each matrix.

$$1. A = \begin{bmatrix} 2 & -1 \\ 0 & 5 \\ -4 & 8 \end{bmatrix}$$

Dimensions: 3x2

$$2. B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Dimensions: 4x1

$$3. C = \begin{bmatrix} 0 & 5 & 3 & -1 \\ -2 & 0 & 9 & 6 \end{bmatrix}$$

Dimensions: 2x4

Different types of Matrices

- **Column Matrix** - a matrix with only one column.
- **Row Matrix** - a matrix with only one row.
- **Square Matrix** - a matrix that has the same number of rows and columns.

Jim, Mario and Mike are married to Shana, Kelly and Lisa. Mario is Kelly's brother and lives in Florida with his wife. Mike is shorter than Lisa's husband. Mike works at a bank. Shana and her husband live in Kentucky. Kelly and her husband work in a candy store. Who is married to whom?

	Jim	Mario	Mike
Shana	X	X	√
Kelly	√	X	X
Lisa	X	√	X

Equal Matrices - two matrices that have the same dimensions and each element of one matrix is equal to the corresponding element of the other matrix.

*The definition of equal matrices can be used to find values when elements of the matrices are algebraic expressions.

Examples: Find the values for x and y

$$1. \begin{bmatrix} 2x \\ 2x + 3y \end{bmatrix} = \begin{bmatrix} y \\ 12 \end{bmatrix}$$

* Since the matrices are equal, the corresponding elements are equal!

$$2x = y$$

* Form two linear equations.

$$2x + 3y = 12$$

* Solve the system using substitution.

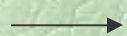
$$\begin{array}{ccc} 2x = y & \longrightarrow & 4y = 12 \\ y + 3y = 12 & & y = 3 \end{array} \qquad \begin{array}{c} 2x = 3 \\ x = \frac{3}{2} \end{array}$$

$$2. \begin{bmatrix} 3x + y \\ x - 2y \end{bmatrix} = \begin{bmatrix} x + 3 \\ y - 2 \end{bmatrix}$$



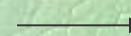
$$3x + y = x + 3$$

$$x - 2y = y - 2$$



$$2x + y = 3$$

$$x - 3y = -2$$



$$2x + y = 3$$

$$\underline{-2x + 6y = 4}$$

$$7y = 7$$

$$y = 1$$

$$2x + 1 = 3$$

$$2x = 2$$

$$x = 1$$



$$x - 2y = y - 2$$

$$1 - 2(1) = 1 - 2$$

$$1 - 2 = -1$$

$$-1 = -1$$



Now check your answer

* Write as linear equations.

* Combine like terms.

* Solve using elimination.

$$3. \begin{bmatrix} 2x & 3 & 3z \end{bmatrix} = \begin{bmatrix} 5 & 3y & 9 \end{bmatrix}$$

Set each element equal and solve!

$$2x = 5$$

$$x = \frac{5}{2}$$

$$3 = 3y$$

$$y = 1$$

$$3z = 9$$

$$z = 3$$

DEFINITION

Matrix (matrices)

Column 1 Column 2 Column 3 Column 4

Row 1

a_{11}

a_{12}

a_{13}

a_{14}

Row 2

a_{21}

a_{22}

a_{23}

a_{24}

Row 3

a_{31}

a_{32}

a_{33}

a_{34}

Row m

a_{mn}

a_{mn}

a_{mn}

a_{mn}

A matrix of m rows and n columns is called a matrix with dimensions $m \times n$.

Example: Find the dimensions.

$$1.) \begin{bmatrix} 2 & -3 & 4 \\ -1 & \frac{1}{2} & \pi \end{bmatrix}$$

2×3

$$2.) \begin{bmatrix} -3 & 8 & 9 \\ \pi & -2 & 5 \\ -6 & 7 & 8 \end{bmatrix}$$

3×3

$$3.) \begin{bmatrix} 10 \\ -7 \end{bmatrix}$$

2×1

$$4.) \begin{bmatrix} -3 & 4 \end{bmatrix}$$

1×2

PRACTICE: Find the dimensions.

$$1.) \begin{bmatrix} -3 & 5 \\ 4 & \frac{1}{4} \\ -\pi & 0 \end{bmatrix}$$

3 X 2

$$2.) \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}$$

2 X 2

$$3.) \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

3 X 3

$$4.) \begin{bmatrix} \pi & \sqrt{2} \end{bmatrix}$$

1 X 2

$$5.) \begin{bmatrix} -5 \\ \pi \end{bmatrix}$$

2 X 1

$$6.) \begin{bmatrix} -3 \end{bmatrix}$$

1 X 1

ADDITION and SUBTRACTION of MATRICES

To add matrices, we add the corresponding elements. **They must have the same dimensions.**

$$A = \begin{bmatrix} -5 & 0 \\ 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} -5 + 6 & 0 + -3 \\ 4 + 2 & 1 + 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 6 & 4 \end{bmatrix}$$

$$2.) \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

When a zero matrix is added to another matrix of the same dimension, that same matrix is obtained.

To subtract matrices, we subtract the corresponding elements. **The matrices must have the same dimensions.**

$$3.) \begin{bmatrix} 1 & 2 \\ -2 & 0 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 3 \\ 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1-1 & 2-(-1) \\ -2-1 & 0-3 \\ -3-2 & -1-3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & -3 \\ -5 & -4 \end{bmatrix}$$

PRACTICE PROBLEMS:

$$1.) \begin{bmatrix} 4 & -1 \\ 6 & -3 \end{bmatrix} + \begin{bmatrix} -6 & -5 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -6 \\ 13 & 0 \end{bmatrix}$$

$$2.) \begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 5 \\ 6 & 4 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -7 \\ -2 & -4 & 8 \end{bmatrix}$$

ADDITIVE INVERSE OF A MATRIX:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 5 \end{bmatrix}$$

$$-A = \begin{bmatrix} -1 & 0 & -2 \\ -3 & 1 & -5 \end{bmatrix}$$

Find the additive inverse:

$$\begin{bmatrix} 2 & -1 & 5 \\ 6 & 4 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -5 \\ -6 & -4 & 3 \end{bmatrix}$$

Scalar Multiplication:

$$k \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 4 & 5 & -6 \end{bmatrix} = \begin{bmatrix} 1k & 2k & 3k \\ -1k & -2k & -3k \\ 4k & 5k & -6k \end{bmatrix}$$

We multiply each # inside our matrix by k.

Examples:

$$1.) \quad 3 \begin{bmatrix} -3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 12 & 15 \end{bmatrix}$$

$$2.) \quad 5 \begin{bmatrix} 1 & -2 & x \\ 4 & y & 1 \\ 0 & -5 & x^2 \end{bmatrix} = \begin{bmatrix} 5 & -10 & 5x \\ 20 & 5y & 5 \\ 0 & -25 & 5x^2 \end{bmatrix}$$

This lesson will prepare you to use matrices to solve problems, including how to find the area of a triangle with vertices anywhere in the coordinate plane.

Essential Understanding The product of a matrix and its *multiplicative inverse matrix* is the *multiplicative identity matrix*. Not all matrices have inverse matrices.

A **square matrix** is a matrix with the same number of rows and columns. While there is a multiplicative identity matrix for any square matrix, not all square matrices have multiplicative inverses.

take note

Key Concepts Identity and Multiplicative Inverse Matrices

For an $n \times n$ matrix, the **multiplicative identity matrix** is an $n \times n$ matrix I , or I_n , with 1's along the main diagonal and 0's elsewhere.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and so forth.}$$

If A and B are square matrices and $AB = BA = I$, then B is the **multiplicative inverse matrix** of A , written A^{-1} .

Problem 1 Determining Whether Matrices are Inverses

For each of the following, are matrices A and B inverses?

A $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Since $AB = I$ and $BA = I$,
matrices A and B are inverses.

TI-84 Plus calculator screen showing matrix multiplication. The top row shows $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The bottom row shows $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The screen also displays '1.a' and 'RAD AUTO REAL'.

B $A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & -2 & 2 \\ -3 & -4 & 5 \\ 1 & 1 & -1 \end{bmatrix}$

Since $AB = I$ and $BA = I$,
matrices A and B are inverses.

TI-84 Plus calculator screen showing matrix multiplication. The top row shows $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & -2 & 2 \\ -3 & -4 & 5 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The bottom row shows $\begin{bmatrix} -1 & -2 & 2 \\ -3 & -4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. The screen also displays '1.b' and 'RAD AUTO REAL'.

How to Determine if Two Matrices are Inverses

1. Multiply the two matrices: AB and BA .
2. If the result is an identity matrix, then the matrices are inverses.

Example: Are A and B inverses?

$$A = \begin{pmatrix} 10 & -3 \\ 3 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 \\ 3 & -10 \end{pmatrix}$$

No, their product does not equal the 2×2 identity matrix

Are C and D inverses?

$$C = \begin{pmatrix} 11 & 2 & -8 \\ 4 & 1 & -3 \\ -8 & -1 & 6 \end{pmatrix} \quad D = \begin{pmatrix} 3 & -4 & 2 \\ 0 & 2 & 1 \\ 4 & -5 & 3 \end{pmatrix}$$

Yes, their product equals the 3x3 identity matrix

Inverse of a Matrix

Multiplicative Inverse of a Matrix

For a square matrix $\mathbf{A}$, the inverse is written $\mathbf{A}^{-1}$. When $\mathbf{A}$ is multiplied by $\mathbf{A}^{-1}$ the result is the identity matrix $\mathbf{I}$.
Non-square matrices do not have inverses.

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Example:

For matrix A , its inverse is A^{-1}

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

Since

$$AA^{-1} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Requirements to have an Inverse

1. The matrix must be square
(same number of rows and columns).
2. The determinant of the matrix must not be zero

A square matrix that has an inverse is called **invertible** or **non-singular**. A matrix that does not have an inverse is called **singular**.

A matrix does not have to have an inverse,
but if it does, the inverse is unique.

Can we find a matrix to multiply the first matrix by to get the identity?

$$\begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let A be an $n \times n$ matrix. If there exists a matrix B such that $AB = BA = I$ then we call this matrix the **inverse** of A and denote it A^{-1} .

$$A = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7 & 3 \\ -2 & -1 \end{bmatrix}$$

Check this answer by multiplying. We should get the identity matrix if we've found the inverse.

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \cdot & a_{mn} \end{bmatrix}, A^T = \begin{bmatrix} a_{11} & a_{21} & \cdot & a_{m1} \\ a_{12} & a_{22} & \cdot & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \cdot & a_{mn} \end{bmatrix}$$

$$\text{Property: } (AB)^T = B^T A^T$$

Symmetric Matrices

$$A=A^T \quad (a_{ij}=a_{ji})$$

Example:

$$\begin{bmatrix} 4 & 5 & -3 \\ 5 & 7 & 2 \\ -3 & 2 & 10 \end{bmatrix}$$

Matrix Inverse

↪ The inverse A^{-1} of a matrix A has the property:

$$AA^{-1}=A^{-1}A=I$$

↪ A^{-1} exists only if $\det(A) \neq 0$

↪ Terminology

- **Singular matrix:** A^{-1} does not exist
- **Ill-conditioned matrix:** A is close to being singular

Matrix Inverse (cont'd)

➤ Properties of the inverse:

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$

Orthonormal matrices

- A is orthonormal if:

$$(1) u_k \cdot u_k = 1 \text{ or } \|u_k\| = 1, \text{ for every } k$$

$$(2) u_j \cdot u_k = 0, \text{ for every } j \neq k \text{ (} u_j \text{ is perpendicular to } u_k \text{)}$$

- Note that if A is orthonormal, it is easy to find its inverse:

$$AA^T = A^T A = I \quad (\text{i.e., } A^{-1} = A^T)$$

Property: $\|Av\| = \|v\|$ (does not change the magnitude of v)

Matrix Multiplication Order

↪ A is n by k , B is k by m

↪ C = A x B defined by

$$c_{ij} = \sum_{l=1}^k a_{il} b_{lj}$$

↪ BxA not necessarily equal to Ax B

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}$$

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} \cdot & * \\ \cdot & * \\ \cdot & * \\ \cdot & * \end{pmatrix}$$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ * & * & * & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \cdot & * \\ \cdot & * \\ \cdot & * \end{pmatrix}$$

Example Multiplications

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} _ & _ \\ _ & _ \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & 3 \\ -3 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} _ & _ & _ \\ _ & _ & _ \\ _ & _ & _ \end{pmatrix}$$

Inverse

↻ If $A \times B = I$ and $B \times A = I$ then

$$A = B^{-1} \text{ and } B = A^{-1}$$