

Module 1 System of Particles

This module is designed to help you to:

- describe the motion of two particles and many particle systems;
- explain the center of mass of a body with illustrations;
- recognize that the position of center of mass of the system depends on the distribution of its mass;
- infer that the position of the center of mass is independent of the coordinate system used;
- apply the concept of center of mass to various systems;
- explain that the motion of a two particle system can be reduced to that of a single particle if we use the center of mass coordinates of the system;
- calculate the loss in kinetic energy in a collision of two particles.

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7.1.1 Introduction

We have seen that Newton's laws of motion are directly applicable for point masses. For applying them to extended objects, we need to think of ways and means of treating them as point objects. Then, the entire mass of the body can be considered to be concentrated at a point. For example, when we handle the two-body interactions, like between the sun and the earth, each is treated as a point mass. One such point is the Centre of Mass (CM) where the whole of the mass of the body is supposed to be concentrated. Newton's laws can then be applied for extended objects considering their mass to be located at their Centre of Mass.

We shall first consider a two-body system and use it to define the Centre of Mass (CM) coordinates. We shall study the advantage of using CM and relative coordinates and thereby define a quantity called the reduced mass. We shall use these for solving problems involving two bodies. Thereafter, we shall extend the concept to many-particle systems.

Next, we shall take up rotational motion. As a prelude to that we shall introduce kinematics of angular motion. There we shall distinguish between uniform and non-uniform angular displacement, angular velocity and angular acceleration. It will provide us the launching pad for studying dynamics of rotational motion, particularly of extended bodies. A couple and its moment, also called torque, are very significant concepts in the study of rotational motion. We shall apply them to the study of rotational motion of extended bodies and relate torque with angular momentum.

In the process, we shall come across the terms Moment of Inertia (MI) and Radius of Gyration. We shall relate torque with angular momentum and see many of its applications including the formulation of Kepler's laws, the model of an atom. We shall also learn other applications, such as rolling of a body with and without slipping. Analogous quantities between linear and rotational motion will be an important feature of this study. Equilibrium of bodies will also be taken up using the knowledge of net couple and force.

7.1.2 The two-body (or two-particle) system and the Center of Mass (CM) coordinates

In the study of physics, we come across many two-body systems like the Sun and the earth or the moon and the earth. At the micro level, we have the case of the motion of an electron around the nucleus. The electron and the

nucleus can be conceived as a two particle-system. Again going by the large distance between the Sun and the earth, or between the earth and the moon, these celestial bodies can be treated as point masses, irrespective of their large radii. So, it would be worthwhile to discuss the role of the centre of mass coordinates in describing the motion of a two-particle system, particularly when the motion is taking place under mutual interaction. In the case of the Sun-earth or earth-moon system it is the mutual force of gravitation and in the case of the electron and the nucleus, it is the mutual electrostatic attraction.

Let the two particles 1 and 2 be located at A and B (Fig. 7.1.1).

Their position with respect to an origin O are, say, $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Let the masses of the particle be m_1 and m_2 .

The equations of motion of the particles are:

$$[\text{Note: } \dot{x} = \frac{dx}{dt} \quad \text{and} \quad \ddot{x} = \frac{d^2x}{dt^2}]$$

$$\mathbf{F}_{12} = m_1 \ddot{\mathbf{r}}_1, \quad (7.1.1)$$

$$\text{and } \mathbf{F}_{21} = m_2 \ddot{\mathbf{r}}_2 \quad (7.1.2)$$

where $\mathbf{F}_{12}$ is the force experienced by particle 1 due to particle 2 and $\mathbf{F}_{21}$ is the force experienced by 2 due to 1.

Since these are forces of mutual attraction, therefore,

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \quad (7.1.3)$$

Using Eq(7.1.3), and adding Eqs(7.1.1) and (7.1.2), we get

$$m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2 = 0 \quad (7.1.4)$$

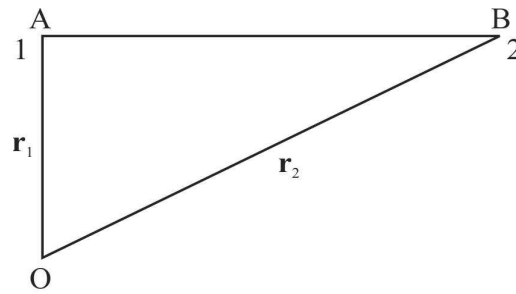


Fig.7.1.1: The two particle system, with particle at A and B

$$\text{or } \frac{d}{dt}(m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2) = 0 \quad (7.1.5)$$

The position vector of the centre of mass is defined as

$$\mathbf{r}_{CM} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \quad (7.1.6)$$

$$\text{Therefore, } \dot{\mathbf{r}}_{CM} = \frac{d}{dt}(\mathbf{r}_{CM}) = \frac{m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2}{m_1 + m_2}$$

It is the velocity of centre of mass of the system, So,

$$(m_1 + m_2) \dot{\mathbf{r}}_{CM} = m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2$$

From Eq. (7.1.5),

$$\frac{d}{dt}\{(m_1 + m_2) \dot{\mathbf{r}}_{CM}\} = \frac{d}{dt}(m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2) = 0$$

$$\text{or, } \frac{d}{dt}(\dot{\mathbf{r}}_{CM}) = 0 \quad (7.1.7)$$

$$\text{or, } a_{CM} = 0 \quad (7.1.8)$$

Thus, the acceleration of the centre of mass is zero and $\dot{\mathbf{r}}_{CM} = \mathbf{a}$, a constant. Hence, the centre of mass of a two-body system drifts in space with a uniform velocity.

Now, we define a vector

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \quad (7.1.9)$$

It is the position vector of particle 2 with respect to 1 or simply the relative vector or the relative coordinate of particle 2 with respect to 1.

The relative velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1 \quad (7.1.10)$

And the relative acceleration, $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 \quad (7.1.11)$

From Eqs. (7.1.1) and (7.1.2),

$$\ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = \frac{\mathbf{F}_{21}}{m_2} - \frac{\mathbf{F}_{12}}{m_1}$$

$\therefore$ From Eqs(7.1.3) and (7.1.11),

$$\mathbf{a} = \mathbf{F}_{21} \left(\frac{1}{m_2} + \frac{1}{m_1} \right) \quad (7.1.12)$$

Or, $\mathbf{F}_{21} = \mu \mathbf{a} = \mu \ddot{\mathbf{r}} \quad (7.1.13)$

where, $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

Or, $\mu = \frac{m_1 m_2}{m_1 + m_2} \quad (7.1.14)$

μ is called the reduced mass of the system.

The significance of the above exercise is that by the introduction of the centre of mass and relative coordinates, we have been able to reduce the two differential equations of motion, Eq(7.1.11) and Eq(7.1.12) to a single equation Eq(7.1.13). This accounts for the motion of the reduced mass and that the effective coordinates involved are the relative coordinates. The coordinates of the centre of mass do not contribute effectively to any equation of motion as its velocity becomes constant Eq(7.1.7).

7.1.3 Centre of Gravity (CG) and Centre of Mass (CM)

Centre of gravity (CG) can be defined only when there is a field of gravity. But the centre of mass is a property of the body and does not depend on external conditions. It depends on the mass distribution of the body only. The centre of gravity coincides with the centre of mass for small bodies as the gravitational field may be taken as uniform over the small dimensions of the body. But the situation may be different when the dimension of the body is large and the gravitational field is not uniform over all parts of the body.

Let us assume we can cut out a tall uniform stone cylinder standing vertically along the peak of Mount Everest. If we now try to locate the CG and the CM of this about 9 km tall cylinder, the two centres will have very little difference. Since the cylinder is uniform the CM will be at the midpoint vertically up. Field of gravity will not be the same near the ground and near the top of the stone cylinder. The lower half of the cylinder will feel slightly stronger gravitational pull compared to that for the upper half. As a result, the CG will get slightly shifted below the midpoint of this vertical cylinder. But once again it needs to be mentioned that with the dimensions of the bodies we consider for our work the CM and CG are two identical points.



Let Us Reflect

Q.7.1.1 Distinguish between centre of gravity and centre of mass.

Q.7.1.2 How can a ring be balanced if its centre of gravity lies outside?



Answers

A.7.1.1 Centre of mass is the point at which the distribution of mass is equal in all directions, and does not depend on the gravitational field. Centre of gravity is the point at which the distribution of weight is equal in all directions, and does depend on the gravitational field.

A.7.1.2 The centre of gravity of a uniform circular ring lies at its geometric centre. Although it lies outside the mass, the centre of gravity of a circular ring balances itself when placed on another object, as shown in Fig. 7.1.2. The gravitational force acts through the supporting object.

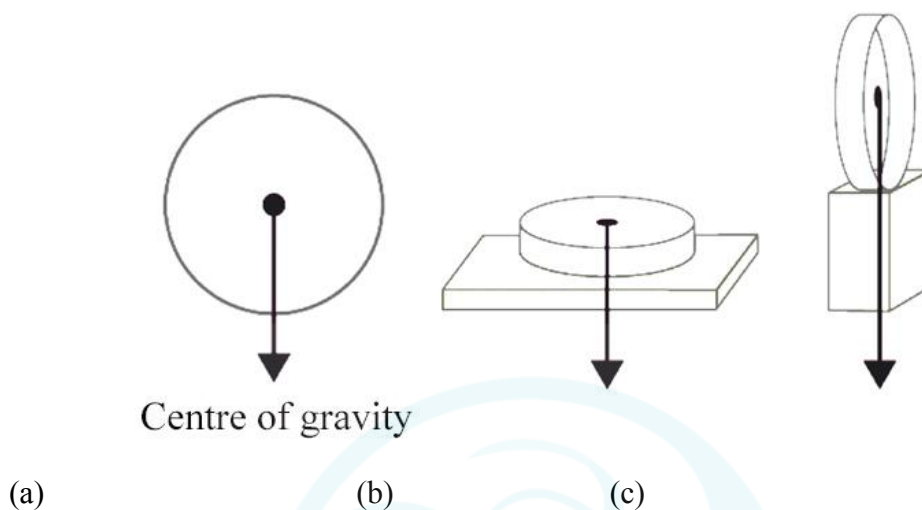


Fig. 7.1.2

7.1.4 Some representative examples of two-body systems

Example 7.1.1 A bullet of mass m moving with a velocity u strikes a block of mass ' M ' which is at rest and free to move in the direction of motion of the bullet and gets embedded into it. Show that the loss of kinetic energy (KE) is $\frac{1}{2} \frac{mM}{m+M} u^2$.

Solution:

Here the two bodies are the bullet (mass = m) and the block (mass = M).

Let the velocity of the combined system, bullet embedded in the block be u .

$$\begin{aligned} \text{So, the velocity of the centre of mass before the strike} &= \frac{m \cdot u + M \cdot 0}{m + M} \\ &= \frac{mu}{m + M} \end{aligned}$$

And velocity of the centre of mass after the strike = v ,

$$\therefore \frac{mu}{m + M} = v$$

[We will also arrive at the same equation by applying the principle of conservation of linear momentum]

$$\text{KE before strike} = \frac{1}{2}mu^2 + \frac{1}{2}M \cdot 0^2 = \frac{1}{2}mu^2$$

$$\text{KE after strike} = \frac{1}{2}(M + m)\left(\frac{mu}{M+m}\right)^2 = \frac{1}{2}\frac{m^2u^2}{M+m}$$

$$\begin{aligned}\therefore \text{Loss of KE} &= \frac{1}{2}mu^2 - \frac{1}{2}\frac{m^2u^2}{M+m} \\ &= \frac{1}{2}mu^2\left(1 - \frac{m}{M+m}\right) = \frac{1}{2}\frac{Mm}{M+m}u^2\end{aligned}$$

$$\text{Now, } \frac{Mm}{M+m} = \mu \quad (\text{The reduced mass of the system})$$

$$\therefore \text{Loss in KE} = \frac{1}{2}\mu u^2$$

So, it is quite interesting to note that the loss in KE is equal to the KE of the reduced mass of the system moving with the relative velocity of the system before the strike. It may be noted that after the strike the relative velocity of m and M is zero.

Example 7.1.2 Two masses m_1 and m_2 , moving along the same straight line, collide and form one body. If a force applied to stop the masses individually before collision would bring them to rest in distances x_1 and x_2 respectively, show that the distance in which the joint body after collision be brought to rest by the same force is given by $\frac{1}{m_1+m_2}\left\{\left(m_1x_1\right)^{\frac{1}{2}} + \left(m_2x_2\right)^{\frac{1}{2}}\right\}^2$.

Solution:

Let the individual velocities of m_1 and m_2 before collision be u_1 and u_2 respectively.

$$\text{So, the velocity of their centre of mass} = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$

After collision they become a single body of mass $(m_1 + m_2)$. At the moment of collision, its velocity, v would be given by

$$v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} \quad (7.1.15)$$

which we get by using conservation of momentum. If the force applied in each case for stopping a body is

$$F, \text{ then } \frac{1}{2}m_1u_1^2 = Fx_1$$

$$u_1^2 = \frac{2F}{m_1} x_1 \quad (7.1.16)$$

$$\text{Similarly, } u_2^2 = \frac{2F}{m_2} x_2 \quad (7.1.17)$$

$$\text{and } v^2 = \frac{2F}{m_1 + m_2} x \quad (7.1.18)$$

where x is the required distance. Comparing Eqs(7.1.15) and (7.1.18), we get

$$\begin{aligned} x &= (m_1 + m_2) \frac{v^2}{2F} \\ \therefore x &= \left(\frac{m_1 + m_2}{2F} \right) \left(\frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} \right)^2 \\ \text{or, } x &= \frac{1}{m_1 + m_2} \left(m_1 \frac{u_1}{\sqrt{2F}} + m_2 \frac{u_2}{\sqrt{2F}} \right)^2 \quad (7.1.19) \end{aligned}$$

From (7.1.16) and (7.1.17), we get

$$\begin{aligned} \frac{u_1^2}{2F} &= \frac{x_1}{m_1} \\ \text{and } \frac{u_2^2}{2F} &= \frac{x_2}{m_2} \\ \therefore \frac{u_1}{\sqrt{2F}} &= \sqrt{\frac{x_1}{m_1}} \\ \text{and } \frac{u_2}{\sqrt{2F}} &= \sqrt{\frac{x_2}{m_2}} \end{aligned}$$

Therefore, from (7.1.19), we get

$$\begin{aligned} x &= \frac{1}{m_1 + m_2} \left(m_1 \sqrt{\frac{x_1}{m_1}} + m_2 \sqrt{\frac{x_2}{m_2}} \right)^2 \\ \text{or, } x &= \frac{1}{m_1 + m_2} \left\{ \sqrt{(m_1 x_1)} + \sqrt{(m_2 x_2)} \right\}^2 \end{aligned}$$

Example 7.1.3 A shell of mass m is ejected from a gun of mass M by an explosion which generates a kinetic energy, E . Prove that the initial velocity of the shell is $\left\{ \frac{2ME}{(M+m)m} \right\}^{\frac{1}{2}}$

Solution:

Let the initial velocity of the shell be v and the velocity of recoil of the gun be V , they are in opposite directions. So the velocity of the centre of mass after explosion = $\frac{mv - MV}{m + M}$

But, the initial velocity of the centre of mass of the shell and the gun is zero.

$$\therefore \frac{mv - MV}{m + M} = 0$$

Negative sign appears because v and V are in opposite directions.

$$\text{or, } mv - MV = 0$$

$$\text{or, } mv = MV \quad (7.1.20)$$

$$\text{or, } V = \frac{mv}{M}$$

$$\text{KE}_{\text{generated}} = E$$

$$\therefore \frac{1}{2}mv^2 + \frac{1}{2}MV^2 = E$$

$$\text{or, } \frac{1}{2}mv^2 + \frac{1}{2}M\frac{m^2v^2}{M^2} = E$$

$$\text{or, } \frac{1}{2}mv^2\left(1 + \frac{m}{M}\right) = E$$

$$\text{or, } v^2 = \frac{2ME}{m(m+M)}$$

$$\text{or, } v = \left\{ \frac{2ME}{(M+m)m} \right\}^{\frac{1}{2}}$$

An analysis of the result

For a mass m moving with velocity v , the linear momentum, $p = mv$

And the KE $E_0 = \frac{1}{2}mv^2$

$$\therefore 2mE_0 = m^2v^2 = p^2$$

$$\text{or, } p = \sqrt{2mE_0} \quad (7.1.21)$$

In a system of two masses m and M ,

$$p = mv = \left\{ 2 \left(\frac{mM}{m+M} \right) E \right\}^{\frac{1}{2}}$$

substituting, $v = \left\{ \frac{2ME}{(M+m)m} \right\}^{\frac{1}{2}}$

$$p = \sqrt{2\mu E}, \quad (7.1.22)$$

where μ is the reduced mass of the system and E its combined KE. In the process of the explosion, there is no external force that acts on the system, so the linear momentum remains conserved.

It is interesting to note from the right hand sides of Eq(7.1.21) and Eq(7.1.22) that the expression of linear momentum retains the same form, but for the combined system we have under the radical sign, the combined KE along with the reduced mass of the system.

Example 7.1.4 A body moving along a straight line, splits into two parts of masses m_1 and m_2 by an internal explosion which generates kinetic energy E . Show that, if after explosion the parts move in the same line as

before, their relative speed is $\sqrt{2E\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}$.

Solution:

Let the velocities of the two parts after explosion be respectively, v_1 and v_2 . Just after the explosion, (using the principle of conservation of linear momentum).

$$m_1 v_1 = m_2 v_2 \quad (7.1.23)$$

$$\text{and } E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \quad (7.1.24)$$

Since, the parts move in opposite direction, their relative velocity v , is given by

$$v = v_1 + v_2 \quad (7.1.25)$$

From Eq (7.1.23), we get

$$\frac{m_1}{v_2} = \frac{m_2}{v_1} = \frac{m_1+m_2}{v_1+v_2} \quad (7.1.26)$$

Again from Eq(7.1.23) and Eq(7.1.24) we get,

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 \left(\frac{m_1 v_1}{m_2} \right)^2$$

$$\begin{aligned}
 &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2} \frac{m_1^2 v_1^2}{m_2} \\
 &= \frac{1}{2}m_1 v_1^2 \left(1 + \frac{m_1}{m_2}\right) \\
 &= \frac{1}{2}m_1 v_1^2 \left(\frac{m_1 + m_2}{m_2}\right) \\
 &= \frac{1}{2}m_1 v_1^2 \left(\frac{v_1 + v_2}{v_1}\right)
 \end{aligned} \tag{7.1.27}$$

From Eq(7.1.25), we get, $E = \frac{1}{2}m_1 v_1 v$

$$\therefore v = \frac{2E}{m_1 v_1} \tag{7.1.28}$$

But from Eq(7.1.27) we get,

$$\begin{aligned}
 v_1^2 &= \frac{2Em_2}{m_1(m_1 + m_2)} \\
 \text{or, } v_1 &= \sqrt{\frac{2Em_2}{m_1(m_1 + m_2)}}
 \end{aligned}$$

Putting this in Eq(7.1.28) we get,

$$\begin{aligned}
 v &= \frac{2E}{m_1} \sqrt{\frac{m_1(m_1 + m_2)}{2Em_2}} \\
 &= \sqrt{\frac{2E(m_1 + m_2)}{m_1 m_2}} \\
 &= \sqrt{2E \left(\frac{1}{m_1} + \frac{1}{m_2} \right)}
 \end{aligned} \tag{7.1.29}$$

Note:- We can also adopt an alternative method, based on the concept of reduced mass, which is as under.

E = the kinetic energy of the two-body system. It can be expressed as $\frac{1}{2}\mu v^2$, where μ is the reduced mass and

v is the relative velocity.

$$\therefore E = \frac{1}{2}\mu v^2$$

$$v = \sqrt{\frac{2E}{\mu}}$$

$$= \sqrt{2E\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}$$

Example 7.1.5 Consider a system of particles in which motion of the particles is referred to the centre of mass of the system.

Show that

$$(a) p_i = p'_i + v'_i$$

where p_i is the momentum of the i^{th} particle (of mass m_i) and $p'_i = m_i v'_i$ where v'_i is the velocity of the i^{th} particle relative to the centre of mass.

Also, prove using the definition of the centre of mass that $\sum p'_i = 0$

(b) Show that $K = K' + \frac{1}{2}MV^2$ where K is the total kinetic energy of the system of particles. K' is the total kinetic energy of the system when the particle velocities are taken with respect to the centre of mass and $\frac{1}{2}MV^2$ is the kinetic energy of the translation of the system as a whole (that is of the centre of mass motion of the system).

Solution: Let us consider a two-particle system.

(a)

$$p = m_1 v_1 + m_2 v_2 \quad (7.1.30)$$

$$v_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \quad (7.1.31)$$

From Eq(7.1.30) on adding and subtracting $(m_1 + m_2) v$ we get,

$$p = m_1(v_1 - v_{CM}) + m_2(v_2 - v_{CM}) + v_{CM}(m_1 + m_2)$$

$$\text{or, } p = m_1 v'_1 + m_2 v'_2 + m v_{CM} \quad (7.1.32)$$

$$\text{And } p_1 = m_1 v_1$$

$$p_1 = m_1(v_1 - v_{CM}) + m_1 v_{CM}$$

$$p_1 = m_1 v'_1 + m_1 v_{CM}$$

$$\text{or, } p_1 = p'_1 + m_1 v \quad (7.1.33)$$

$$\text{Similarly, } p_2 = p'_2 + m_2 v \quad (7.1.34)$$

Again, from Eq(7.1.32) we get

$$mv_{CM} = p'_1 + p'_2 + mv_{CM}$$

$$\therefore p'_1 + p'_2 = 0 \quad (7.1.35)$$

(b) Total Kinetic Energy of System

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_1 \cdot v_1 + \frac{1}{2}m_2 v_2 \cdot v_2$$

$$K = \frac{1}{2}m_1(v'_1 + v_{CM})(v'_1 + v_{CM}) + \frac{1}{2}m_2(v'_2 + v_{CM})(v'_2 + v_{CM})$$

$$K = \left(\frac{1}{2}m_1 v'^2_1 + \frac{1}{2}m_2 v'^2_2\right) + \frac{1}{2}(m_1 + m_2)v_{CM}^2 + m_1 v'_1 v_{CM} + m_2 v'_2 v_{CM}$$

$$K = K' + \frac{1}{2}MV^2 + (m_1 v'_1 + m_2 v'_2)v_{CM}$$

The quantity in bracket in the third term = $p'_1 + p'_2 = 0$

by Eq(7.1.35),

$$K = K' + \frac{1}{2}MV^2$$

Discussion on the centre of mass coordinates:

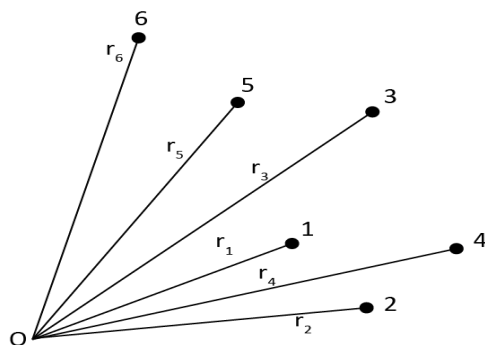


Fig.7.1.3 Many particle system

The position of the center of mass of a two-body system is given by

$$r_{CM} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

The same for a many particle system (Fig. 7.1.3) is given by

$$r_{CM} = \frac{m_1 r_1 + m_2 r_2 + \dots + m_n r_n}{m_1 + m_2 + \dots + m_n}$$

$$\therefore r_{CM} = \frac{\sum_{i=1}^n m_i r_i}{\sum_{i=1}^n m_i} \quad (7.1.36a)$$

Or simply we can write for a system of n particles

$$r_{CM} = \frac{\sum m_i r_i}{\sum m_i} \quad (7.1.36b)$$

$$r_{CM} = x_{CM} \hat{i} + y_{CM} \hat{j} + z_{CM} \hat{k}$$

If we break up r_i into x , y , z - components, we get

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \dots + m_n x_n}{m_1 + m_2 + m_3 + \dots + m_n} \quad (7.1.37a)$$

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \dots + m_n y_n}{m_1 + m_2 + m_3 + \dots + m_n} \quad (7.1.37b)$$

$$\text{And, } z_{CM} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + \dots + m_n z_n}{m_1 + m_2 + m_3 + \dots + m_n} \quad (7.1.37c)$$

$$\therefore x_{CM} = \frac{\sum m_i x_i}{\sum m_i} \quad (7.1.38a)$$

$$y_{CM} = \frac{\sum m_i y_i}{\sum m_i} \quad (7.1.38b)$$

$$z_{CM} = \frac{\sum m_i z_i}{\sum m_i} \quad (7.1.38c)$$

If we have a continuous body, then the summation signs get replaced by the integration sign. And we have,

$$x_{CM} = \frac{\int x dm}{\int dm} \quad (7.1.39a)$$

$$y_{CM} = \frac{\int y dm}{\int dm} \quad (7.1.39b)$$

$$z_{CM} = \frac{\int z dm}{\int dm} \quad (7.1.39c)$$

In all the above equations, dm is an elementary mass and integration is taken over the entire body.

It gets reflected through Eqs(7.1.37), (7.1.38) and (7.1.39) that the position of the centre of mass (r_{CM}) depends on the distribution of mass. Now, let us see some more examples.

Example 7.1.6 Five point masses equal to 1 g, 2 g, 3 g, 4 g and 5 g are kept at A, B, C, D and E respectively (Fig 7.1.4) which are points on a straight line, such that, $AB = BC = CD = DE = 1$ cm. Where is the location of the centre of mass? If we redistribute the masses, such that the order is changed with the masses, 5 g, 4 g, 3 g, 2 g and 1 g at A, B, C, D and E respectively, will the centre of mass remain the same?



Fig. 7.1.4

Solution: We consider the straight line AE to be the x axis with the origin at A.

From Eqn. (7.1.37a) we get,

$$x_{CM} = \frac{1 \times 0 + 2 \times 1 + 3 \times 2 + 4 \times 3 + 5 \times 4}{1 + 2 + 3 + 4 + 5} \text{ cm}$$

$$= \frac{40}{15} \text{ cm} = 2.67 \text{ cm}$$

It may be noted that the dimension of the numerator is $[M][L]$ and that of the denominator is $[M]$. So x_{CM} has the dimension of length. It is at a distance of 2.67 cm from A.

x_{CM} is located between C and D.

In the second case,

$$x_{CM} = \frac{5 \times 0 + 4 \times 1 + 3 \times 2 + 2 \times 3 + 1 \times 4}{5 + 4 + 3 + 2 + 1} \text{ cm}$$

$$= 1.33 \text{ cm}$$

So, the centre of mass shifts to a point between B and C. It is now at a distance of 1.33 cm from A.

This example shows that the centre of mass of a system depends on its mass distribution.

Now, let us take an example from a two-dimensional system.

Example 7.1.7 Refer to Fig. 7.1.5(a) and 7.1.5(b). Each side of the square ABCD is 2 cm. Three masses, 1 g, 2 g and 3 g have been placed at three vertices in the manner shown in the figures. Find the position vectors of the centre of mass in the planes of the two squares

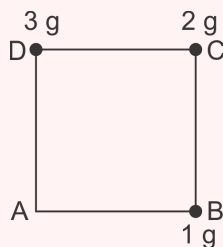


Fig. 7.1.5(a)

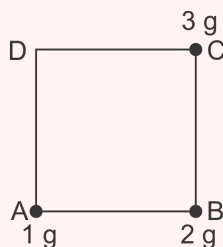


Fig. 7.1.5(b)

Solution:

$$r_{CM} = \frac{m_1 r_1 + m_2 r_2 + m_3 r_3}{m_1 + m_2 + m_3}$$

Let us select the two-dimensional coordinate system with A as the origin, AB and AD as the x and y -axes respectively.

For Fig.7.1.5(a), $r_1 = AB = 2\hat{i}$

$$r_2 = AC = AB + BC = 2\hat{i} + 2\hat{j}$$

$$r_3 = AD = 2\hat{j}$$

$$\begin{aligned} \therefore r_{CM} &= \frac{1(2\hat{i}) + 2(2\hat{i} + 2\hat{j}) + 3 \times 2\hat{j}}{1 + 2 + 3} \\ &= \frac{6\hat{i} + 10\hat{j}}{6} \end{aligned}$$

$$r_{CM} = \frac{1}{3}(3\hat{i} + 5\hat{j}) = (1)\hat{i} + \left(\frac{5}{3}\right)\hat{j}$$

For Fig.7.1.5(b), $r_1 = 0$

$$r_2 = AB = 2\hat{i}$$

$$r_3 = AC = 2\hat{i} + 2\hat{j}$$

$$\begin{aligned} \therefore r_{CM} &= \frac{1(0) + 2(2\hat{i}) + 3(2\hat{i} + 2\hat{j})}{1 + 2 + 3} \\ &= \frac{10\hat{i} + 6\hat{j}}{6} \end{aligned}$$

$$= \frac{1}{3}(5\hat{i} + 3\hat{j})$$

$$= \left(\frac{5}{3}\right)\hat{i} + (1)\hat{j}$$

We find that the centre of mass shifts from the point $\left(1, \frac{5}{3}\right)$ to $\left(\frac{5}{3}, 1\right)$.

Thus, it has been established through the examples that the centre of mass is not a fixed point. Its position depends on the mass distribution. **For a given distribution of mass, the position of the centre of mass is**

independent of the choice of the origin. This can be established by calculating the position of the centre of mass by taking point B as origin.

Now, let us try to look at a similar situation by taking continuous systems.

Example 7.1.8 (i) To determine the centre of mass of a thin uniform rod, where thickness can be ignored. The length of the rod is l .

(ii) What would happen if the mass distribution varies linearly with the distance from one end?

Solution: (i) Refer to Fig 7.1.6. Let the straight rod be placed along the x -axis, its left end coinciding with the origin of the coordinate system. So the coordinates of the two ends are respectively, $x = 0$ and $x = l$.

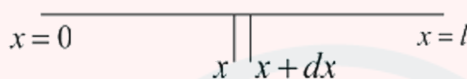


Fig. 7.1.6

We consider an infinitesimal element of length dx between x and $x + dx$.

Let the mass per unit length of the rod = λ

The mass of the element = $\lambda dx = dm$

Therefore, from Eq(7.1.39a), we get

$$x_{CM} = \frac{\int_0^l x \lambda dx}{\int_0^l \lambda dx}$$

$$= \frac{\lambda \int_0^l x dx}{\lambda \int_0^l dx} = \frac{\int_0^l x dx}{\int_0^l dx} = \frac{\frac{l^2}{2}}{l} = \frac{l}{2}$$

So the centre of mass is, as expected, at the midpoint of the rod.

(ii) For the second case, let the proportionality constant for variation of mass distribution with the distance from one end be k , i.e. mass at a distance x from one end taken as origin is kx .

Then, the mass of the element dx is $kx dx$. Therefore,

$$x_{CM} = \frac{\int_0^l x k x dx}{\int_0^l k x dx} = \frac{\int_0^l x^2 dx}{\int_0^l x dx} = \frac{\frac{l^3}{3}}{\frac{l^2}{2}} = \frac{2l}{3}$$

So x_{CM} shifts from $x = \frac{l}{2}$ to $x = \frac{2l}{3}$.

So, the position of the centre of mass depends on the mass distribution. It is the point where the entire mass of the system can be considered to be located. In many situations, we replace a body of mass m by a point mass m . For example, when we consider the gravitational attraction between the sun and the earth, we consider the masses of the sun and the earth to be concentrated at their centres of mass.

Let us suppose that a composite body is being lifted from the floor. After lifting, the lower end of the body is at a height, h_1 and the upper end at a height, h_2 from the ground. By what extent has the body been lifted? It is neither h_1 or h_2 . It is the height of the centre of mass of the body which has been lifted and we calculate the gravitational potential energy of the body with respect to this point.

7.1.5 Loss of kinetic energy in a two-body collision

We have already studied the two-body system, wherein we have discussed several specific cases. Let us now discuss the general case of a two-body system.

Example 7.1.9 A body of mass m_1 traveling in a straight line with a velocity u_1 collides with another body of mass m_2 traveling with a velocity u_2 along the same line such that $u_2 > u_1$. They form a single body after collision. What is the loss in KE?

Solution: Let the velocity of the combined body after collision be v . Then,

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2) v$$

$$\text{Or, } v = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$$

$$\begin{aligned} \text{KE after collision} &= \frac{1}{2} (m_1 + m_2) \left(\frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} \right)^2 \\ &= \frac{1}{2} \frac{(m_1 u_1 + m_2 u_2)^2}{m_1 + m_2} = K_2, \text{ say} \end{aligned}$$

And KE before collision = $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = K_1$, say

Loss is KE = $K_1 - K_2$

$$K_1 - K_2 = \frac{1}{2} \left\{ m_1u_1^2 + m_2u_2^2 - \frac{(m_1u_1 + m_2u_2)^2}{m_1 + m_2} \right\} \quad (7.1.40)$$

$$\begin{aligned} &= \frac{1}{2} \left\{ m_1m_2 \frac{(u_1^2 + u_2^2 - 2u_1u_2)}{m_1 + m_2} \right\} \\ &= \frac{1}{2} \frac{m_1m_2}{m_1 + m_2} (u_1 - u_2)^2 \quad (7.1.14) \\ &= \frac{1}{2} \mu u^2, \end{aligned}$$

where μ = the reduced mass of the system and u = the relative velocity of the system before collision

So, in a collision of two particles, the loss in KE of is equal to the KE of a mass equal to the reduced mass of the two-body system traveling with a velocity equal to the relative velocity of the system before collision. The situation can be compared to what we had obtained in Example 7.1.1 There expression for loss in KE can also be expressed as $\frac{1}{2}\mu u^2$, where $\mu = \frac{Mm}{M+m}$ = the reduced mass of the system and u = the relative velocity before collision because the block had been at rest before being struck by the bullet traveling with velocity u .

It would be advisable to discuss with the students as many problems as possible related to the two-body system and apply it to cases such as the earth-moon system, earth-sun system or a system of binary stars. It can also be used to describe the motion of the proton-electron system. It is interesting to note the following: If the position vectors of masses m_1 and m_2 are respectively r_1 and r_2 , then the position vector of the centre of mass is

$$r_{CM} = \frac{m_1r_1 + m_2r_2}{m_1 + m_2} \text{ and}$$

$$\mu = \frac{m_1m_2}{m_1 + m_2}$$

$$\text{Thus, } r_{CM} = \frac{r_1 + \frac{m_2}{m_1}r_2}{1 + \frac{m_2}{m_1}} \text{ and}$$

$$\mu = \frac{m_2}{1 + \frac{m_2}{m_1}}$$

$$\text{If } m_1 \gg m_2, \frac{m_2}{m_1} \rightarrow 0$$

and so

$$r_{CM} \rightarrow r_1 \text{ and } \mu \rightarrow m_2$$

Such a situation happens with the sun-earth or proton-electron systems, and the motion can be approximated to that of the motion of the smaller mass about the larger mass, e.g. the earth about the Sun or electron about the proton.



Let Us Reflect

Q.7.1.3 A child of weight 37 kg and a man of weight 80 kg are standing 10 m apart on frictionless ice.

- How far from the child is their center of mass?
- If each holds one end of a rope, and the man pulls on the rope so that he moves 2.0 m, how far from the child will he be now?



Answers

A.7.1.3 mass of child, $m_c = 37$ kg,

mass of man, $m_m = 80$ kg

separation distance between the man and the child, $L = 10$ m

i) Finding the distance of center of mass (x) from the child

Assuming the line joining the man and the child as the x-axis with the child at the origin, and the man at a distance of 10 m, i.e.

$$x_c = 0 \text{ m and } x_m = 10 \text{ m}$$

The position of center of mass from the child is calculated as:

$$\begin{aligned} X &= \frac{m_c x_c + m_m x_m}{m_c + m_m} \\ &= \frac{37 \times 0 + 80 \times 10}{37 + 80} = 6.84 \text{ m} \end{aligned}$$

(ii) Finding the new distance between the man and the child after the man pulls the rope

It is given that the distance moved by the man is 2.0 m, i.e. $dm = 2.0 \text{ m}$

Let the distance moved by the child be $-dw$

(negative sign signifies that child must move in opposite direction of the man)

Since both man and child are standing on ice with no friction, there is no external force. Therefore, the displacement of center of mass must be zero, i.e. $d_{CM} = 0$

Now, the displacement of center of mass is expressed as:

$$d_{cm} = \frac{m_c(-d_c) + m_m(d_m)}{m_c + m_m}$$

$$0 = \frac{37(-d_c) + 80(2)}{37 + 80}$$

$$37 d_c = 160$$

$$d_c = \frac{160}{37} = 4.32 \text{ m}$$

Therefore, the new separation distance between the man and the child is:

$$d = L - d_m - d_c$$

$$= 10 - 2 - 4.32$$

$$= 3.68 \text{ m}$$

7.1.6 Summary

- Motion of a two-body (or two-particle) system can be treated as the motion of the centre of mass of the system and the motion around the centre of mass.
- In general, the Centre of Gravity (CG) and the Centre of Mass (CM) are two different points. These two points are coincident in the case when the gravitational field is uniform.
- The centre of mass of a system depends on its mass distribution and is independent of the choice of origin to which particles are referred.
- The motion of a two particle system can be reduced to that of a single particle located at the centre of mass of the system.
- If the mass of one of the particles in a two particle system is very small compared to the mass of the other particle, then the equation of motion of the system is reduced to that of the lighter particle.
- In a collision of two particles, the loss in KE of is equal to the KE of a mass equal to the reduced mass of the two-body system traveling with a velocity equal to the relative velocity of the system before collision.

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