

Module 2 Kinematics of Rotational Motion

This module is designed to help you to:

- distinguish between linear and angular motion;
- distinguish between uniform and non uniform angular motion;
- derive kinematic equations of angular motion;
- apply the kinematic equations to uniformly accelerated angular motion for solving problems relating to angular motion.

Content

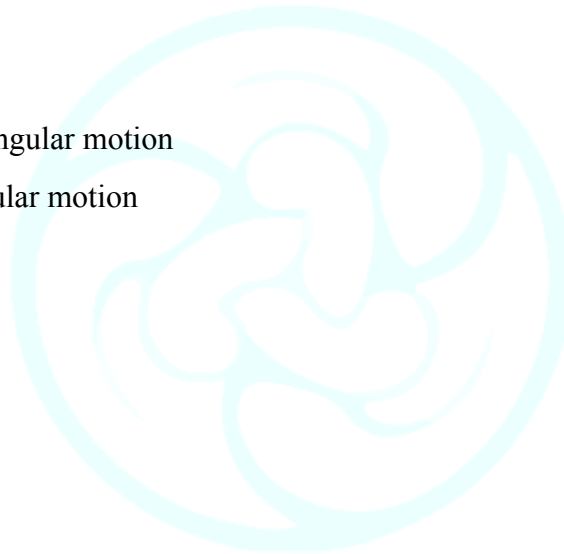
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7.2.1 Introduction

Can you imagine what our world would be like without wheels! Transportation will become a very big problem. The motion of a wheel is the most important example of angular motion. Same is the case of a ceiling fan. All planets execute angular motion around the sun. An athlete taking part in a 200 m race also executes angular motion.

In this module, we shall learn to define several parameters related to angular motion in general and circular motion in particular. We shall learn to distinguish between uniform and non-uniform angular motion. Correspondingly, we shall define angular velocity and angular acceleration, and their relation with linear velocity and linear acceleration.

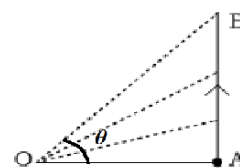
We shall derive the kinematical equations for motion under uniform angular acceleration and illustrate their application by solving some problems.

7.2.2 Angular motion

You have studied linear motion, where our main concern was the analysis of displacement with time. Now, what is an angular motion? A particle moving along a curve is said to execute angular motion. Do you know that linear motion can also be treated as angular motion? We consider the motion of a particle along a straight line AB. Its position is changing continually with reference to the point A (Fig 7.2.1(a)). So, it is a linear motion. But its angular position with reference to the point O and line OA is also changing continuously. When such a situation occurs, it is said that the particle is executing angular motion (Fig 7.2.1(b)). For studying angular motion, we need a reference point (O) and a reference line (OA).



(a) Linear Motion



(b) angular motion

Fig. 7.2.1

7.2.3 Uniform and non-uniform angular motion

The angle described by a particle per unit time is its angular velocity ω . In the notation of differential calculus,

$$\omega = \frac{d\theta}{dt} \quad (7.2.1)$$

Its unit is radian per second (rad s^{-1}).

When angular velocity is constant, the motion of the particle is termed as uniform angular motion.

Example 7.3.1

What are the angular velocities of the hour, the minute and the second hands of a clock?

Solution:

The hour, the minute and the second hands describe $360^\circ (=2\pi \text{ radians})$ in 12 h (i.e. $12 \times 60 \times 60 \text{ sec}$), 60 min ($60 \times 60 \text{ s}$) and 1 min (60 s) respectively. So, the respective angular velocities are,

$$\omega_{\text{hour}} = \frac{2\pi}{12 \times 60 \times 60} = \frac{\pi}{21600} \text{ rad s}^{-1}$$

$$\omega = \frac{2\pi}{60 \times 60} = \frac{\pi}{1800} \text{ rad s}^{-1}$$

$$\omega = \frac{2\pi}{60} = \frac{\pi}{30} \text{ rad s}^{-1}$$

Example 7.2.2 A particle describes a path such that the variation of its angular displacement with reference to time is given by $\theta = At + Bt^2$, where A and B are constants. What is the angular velocity of the particle?

Solution:

Differentiating, $\theta = At + Bt^2$

$$\omega = \frac{d\theta}{dt} = A + 2Bt \quad (7.2.2)$$

From the solution of Eq(7.2.2), we see that ω is a function of t which is indicative of non-uniform angular motion. There are many examples in real life where we come across non-uniform angular motion. For example, the motion of a ceiling fan is non uniform when it is switched 'ON'. It starts from zero angular velocity and attains an angular velocity of a higher value within a certain time. The rate of change of angular velocity with time is called angular acceleration and is given by

$$\alpha = \frac{d\omega}{dt} \quad (7.2.3)$$

$$\text{Or, } \alpha = \frac{d^2\theta}{dt^2} \quad (7.2.4)$$

The unit of angular acceleration is radian per second square (rad s^{-2}).

From Eq(7.2.1),

$$\alpha = 2B$$

Since B is constant, the angular acceleration α is also constant.

Thus, it represents a case of uniform angular acceleration. We shall now study the kinematics of angular motion but would restrict our discussion primarily to uniform angular acceleration

7.2.4 Kinematic equations of angular motion

You have already learnt the kinematics of linear motion. The angular motion is handled in a similar manner by considering the analogy between displacement, velocity, acceleration with angular displacement, angular velocity and angular acceleration.

For a body executing angular motion about an axis, like a fan or a wheel, every point of the body executes circular motion about that axis (Fig. 7.2.2). The angular displacement is measured with reference to a point O

where the axis intersects the plane of motion and a reference line OA. It may be noted that the angular displacement of every point in the body is the same. Here, we are considering only uniformly accelerated motion. Now, let the angular velocity of the point P at any time t be ω .

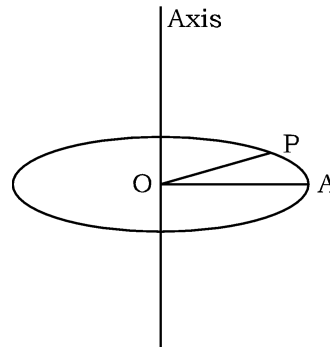


Fig. 7.2.2 : uniformly accelerated motion.

Then, $\frac{d\omega}{dt} = \alpha$, a constant

For uniform angular acceleration,

$$d\omega = \alpha dt \quad (7.2.5)$$

On integrating both sides of Eq(7.2.5), we get

$$\omega = \alpha t + C_1 \quad (7.2.6)$$

where C_1 is a constant.

Let $\omega = \omega_0$ when $t = 0$

$$\therefore \omega_0 = C_1$$

Hence, from Eq(7.2.6), we get

$$\omega = \omega_0 + \alpha t \quad (7.2.7)$$

ω_0 , the angular velocity at, $t = 0$ is referred to as the initial angular velocity.

Again from Eq(7.2.1), we get

$$\frac{d\theta}{dt} = \omega_0 + \alpha t$$

$$\text{or, } d\theta = (\omega_0 + \alpha t)dt$$

On integration, we get

$$\theta = \int \omega_0 dt + \int \alpha t dt$$

$$= \omega_0 t + \frac{\alpha t^2}{2} + C_2 \quad (\omega_0 \text{ and } \alpha \text{ are constants})$$

Hence, C_2 is a constant of integration.

If $\theta = 0$ at $t = 0$, we get $C_2 = 0$

$$\therefore \theta = \omega_0 t + \frac{1}{2}\alpha t^2 \quad (7.2.8)$$

In other words, the angular displacement during a time t with uniform angular acceleration, and initial angular velocity, ω_0 is given by Eq(7.2.7).

From Eq(7.2.7), we get,

$$\omega^2 = (\omega_0 + \alpha t)^2 = \omega_0^2 + 2\omega_0 \alpha t + \alpha^2 t^2 = \omega_0^2 + 2\alpha\left(\omega_0 t + \frac{1}{2}\alpha t^2\right)$$

$\therefore$ From Eq(7.2.8), we get

$$\omega^2 = \omega_0^2 + 2\alpha\theta \quad (7.2.9)$$

Eq(7.2.9) is apparently independent of time, but in effect time is included in θ and ω .

Eqs(7.2.7), (7.2.8) and (7.2.9) are analogous to the kinematic equations of linear motion with uniform acceleration, given by

$$v = u + at$$

$$x = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

Example 7.2.3

Show that the average angular velocity of a body moving with uniform angular acceleration is the arithmetic mean of the initial angular velocity (ω_0) and the final angular velocity (ω) at time t .

Solution:

Average angular velocity during a time t is given by

$$\bar{\omega} = \frac{\text{Angular Displacement}}{t}$$

$$\therefore \bar{\omega} = \frac{1}{t} \left(\omega_0 t + \frac{1}{2} \alpha t^2 \right)$$

From Eq. (7.2.8)

$$\bar{\omega} = \omega_0 + \frac{\alpha t}{2}$$

$$\therefore \bar{\omega} = \frac{\omega_0 + (\omega_0 + \alpha t)}{2}$$

$$\text{Or } \bar{\omega} = \frac{1}{2} (\omega_0 + \omega)$$

Example 7.2.4 A ceiling fan on being switched ‘ON’, attains an angular speed of 300 rev/min in 10 seconds. After that it rotates with uniform angular speed. What is its angular acceleration? How many revolutions does it undergo in the first 1 minute?

Solution:

Here, $\omega_0 = 0 \text{ rad s}^{-1}$

$$\omega = 300 \text{ rev min}^{-1} = 5 \text{ rev s}^{-1} = 10 \pi \text{ rad s}^{-1}$$

From Eq(7.2.8)

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{\omega}{t}$$

$$\therefore \alpha = \frac{10\pi}{10} \text{ rad s}^{-2} = \pi \text{ rad s}^{-2}$$

$$\text{Angle described in } 10 \text{ s} = \frac{1}{2} \alpha 10^2 = \frac{1}{2} \pi 100 = 50 \pi \text{ rad}$$

$$\text{In the next } 50 \text{ s, angle described} = 10 \pi \times 50 = 500 \pi \text{ rad}$$

$$\text{So, total angle described} = (500 + 50) \pi = 550 \pi$$

$$\text{no. of revolutions} = 550 \pi / 2 \pi = 275 \text{ rev}$$

Now that you have learnt about angular acceleration you may be interested in knowing about its relation with linear acceleration. We shall do it for a uniformly accelerated motion along a circle.

We know that the linear velocity is given by

$$v = \omega R, \quad (7.2.10)$$

where R is the radius of the circle.

The linear acceleration, a is given by

$$a = \frac{dv}{dt} = \frac{d}{dt} (\omega R)$$

$$\therefore a = \frac{d\omega}{dt} R$$

$$\text{Or, } a = \alpha R \quad (7.2.11)$$

This was indeed expected.



Let Us Reflect

Q.7.2.1 A particle is executing uniform circular motion. Its linear and angular velocities are v and ω respectively. From a certain position, it undergoes an infinitesimal rotation $\Delta\theta$ during an infinitesimal time Δt . Find the components of the change in linear velocity along and perpendicular to the original direction.

Q.7.2.2 A wheel is rotating with a uniform angular speed of 600 rev min^{-1} . It is brought to rest in one minute. What is angular retardation? How many revolutions did it go through during that one minute?

Q.7.2.3 A particle executes circular motion with uniform acceleration 3 m s^{-2} . What is the angular acceleration if the radius of the circle is 15 cm ?



Answers

A.7.2.1 Refer to Fig. 7.2.3.

Let $\angle AOB = \Delta\theta$

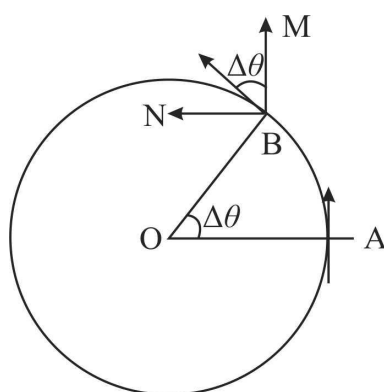


Fig 7.2.3

At B we have chosen two mutually perpendicular directions parallel and perpendicular to the original direction of θ at A. The components of θ , parallel and perpendicular to the original direction are

$$v \cos \Delta\theta, v \sin \Delta\theta$$

As $\Delta\theta \rightarrow 0$, $\cos \Delta\theta \rightarrow 1$ and $\sin \Delta\theta \rightarrow \Delta\theta$

So, the values are v and $v\Delta\theta$

$$\therefore \text{Change along the original direction} = v - v = 0$$

$$\text{The change perpendicular to the original direction} = v\Delta\theta - 0 = v\Delta\theta$$

There is no change in the velocity along the original direction. But a change takes place in a direction perpendicular to it, i.e. towards the centre of the circle. The corresponding value of acceleration

$$a = \frac{v\Delta\theta}{\Delta t}$$

As $\Delta t \rightarrow 0$

$$a = \frac{v d\theta}{dt} = v\omega = \frac{v \cdot v}{R}$$

where R = radius of the circle

$$\therefore a = \frac{v^2}{R}$$

$$\text{A.7.2.2 } n = 600 \text{ rev min}^{-1} = 10 \text{ rev s}^{-1}$$

$$\text{therefore angular speed } \omega = 2\pi n = 20\pi \text{ rad s}^{-1}$$

$$\text{The required retardation } \alpha = \frac{20\pi - 0}{60} \text{ rad s}^{-2} = \frac{\pi}{3} \text{ rad s}^{-2}$$

The angle described during this time

$$\theta = \frac{\omega + \omega_0}{2} \Delta t$$

$$\theta = \frac{20\pi + 0}{2} \times 60$$

We have,

$$\theta = 600 \pi \text{ rad}$$

$$\therefore n = \frac{\theta}{2\pi} = 30 \text{ rev}$$

A.7.2.3 We know that

$$\alpha = \frac{a}{R} = \frac{3 \text{ m s}^{-2}}{0.15 \text{ m}} = 20 \text{ rad s}^{-2}$$

We have discussed the kinematics by considering rotation about an axis. If instead we consider rotation about a point on the axis and if the radius vector of the rotating point is r , the angular velocity ω , the angular acceleration α , then the linear velocity v and acceleration a , then v and a can be shown to be expressed as,

$$v = \omega \times r \quad (7.2.12)$$

$$a = \alpha \times r \quad (7.2.13)$$

7.2.5 Summary

- When the line joining a moving point with a reference point keeps changing its angle with time with respect to a reference line, then it is said to execute angular motion.
- If equal angles are described in equal intervals of time, then the motion is called a uniform angular motion, otherwise it is a non-uniform motion.
- For a body executing angular motion about a fixed axis, every point in it is in circular motion.
- The linear and angular velocities ($\mathbf{v}$ and ω), linear and angular accelerations ($\mathbf{a}$ and α) are related to each other as (where R is radius of the circle)

$$\mathbf{v} = \omega R;$$

$$\mathbf{a} = \alpha R$$

and the general relations are $\omega = \frac{d\theta}{dt}$, $\alpha = \frac{d\omega}{dt}$

For a body rotating about a point, the above relations between $\mathbf{v}, \mathbf{a}, \omega$, and α are represented vectorially with respect of the radius vector $\mathbf{r}$ of rotation as:

$$\mathbf{v} = \omega \times \mathbf{r}$$

$$\mathbf{a} = \alpha \times \mathbf{r}$$

- For uniformly accelerated angular motion, the kinematic equations (with symbols having usual meanings) are

$$\omega = \omega_0 + \alpha t$$

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

The average angular velocity during a certain period is the arithmetic mean between the initial and final angular velocities.



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