

## Module 3 Dynamics of Rotational Motion

### This module is designed to help you to:

- describe the rotational motion of a body as the combination of the motion of its constituent particles;
- **explain** moment of a force (also called torque) and angular momentum for understanding the dynamics of rotational motion;
- relate the lever arm of a torque and angular momentum;
- identify analogous terminologies, expression and equations used in dynamics of rotational motion with that of linear motion;
- apply the principle of conservation of angular momentum in solving problems;
- apply the concept of force and couple in determining the equilibrium condition of bodies.

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### 7.3.1 Introduction

You must have realised by now that the rotational motion of an extended body such as that of a wheel, a fan, a spinning top, a merry-go-round, the earth can be considered as a combined motion of all the particles constituting the body. We have studied earlier the kinematics of angular motion of particles. Now we discuss the dynamics of rotational motion.

As you know, mass and force play important roles in the understanding of the dynamics of linear motion. They continue to play similar roles here, but we shall see that they manifest themselves in different forms. In the process we shall learn about moment of inertia, torque and angular momentum.

First, we shall develop the concept of torque through the idea of moment of a force and then gradually build up using moment of inertia. We shall introduce the meaning of radius of gyration. We shall also get to know about the expressions of moment of inertia of different bodies about different axes. We shall apply these concepts to solve various problems of rotational motion, one of which is rolling with and without slipping.

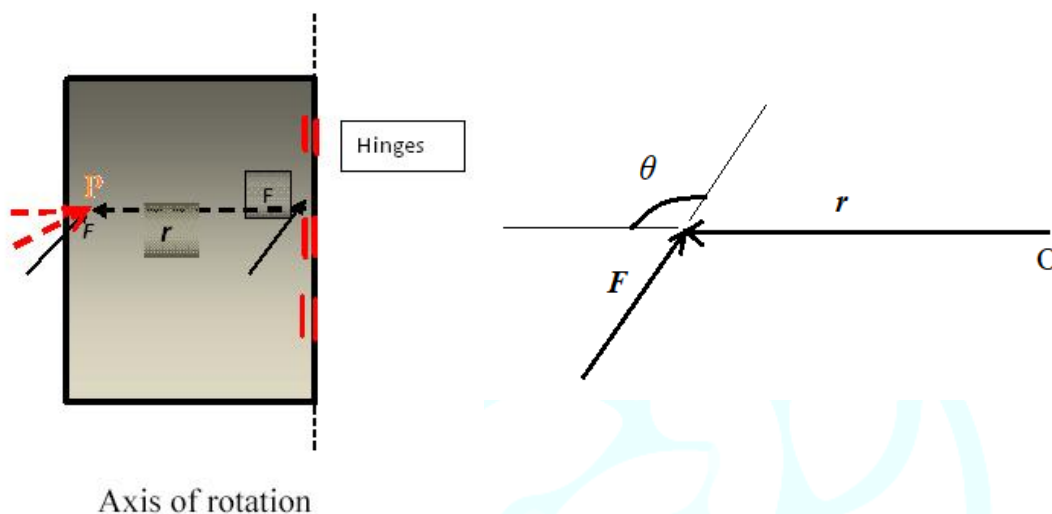
The study of rotational motion combined with that of linear motion will help us in having an understanding about equilibrium of bodies. We shall conclude this module with the study of equilibrium of bodies.

### 7.3.2 Common examples of rotation of bodies

Motion of a wheel, a spinning top, a ceiling fan, the earth are examples of rotational motion. You must have observed that there is always an axis of rotation. Moreover, every particle constituting the body moves in a circle about the same axis with the same angular velocity. For example, let us visualise the rotation of the earth. Take a globe and identify the location of New Delhi on it. This point will lie on a specific latitude, which is a small circle. Like that every place lies on a small circle and those on the equator lie on a great circle. It is important to note that every point on the globe moves in circles of varying radii. The axis of rotation in each case is the polar axis, that is, the line joining the south and the north poles.

Now, let us take a closer look at a very common everyday event. If you want to close a door, then you should be able to locate the axis of rotation (Fig 7.3.1). Assume that a force  $\mathbf{F}$  is applied at point P. Let the radius vector from the axis of rotation to point P be  $\mathbf{r}$ .

If  $\mathbf{r}$  is constant, the force can be applied at different points on the edge of the door. We find that rotation of the door is easier when a stronger force is applied. That may sound quite obvious. We may say that the rotational effect on the door depends upon the magnitude of force  $\mathbf{F}$ .



**Fig. 7.3.1:** Closing a door applying rotation

**Inference I:** Rotational effect is proportional to the magnitude of the applied force.

You may push a door at various points on the door. At what distance from the axis is the closing/ opening of the door easier?

It is found that if the distance between the point on the door where the force is applied and the axis of rotation is larger, rotation of the door is easier even if the same force is applied. If the distance is gradually decreased, closing of the door becomes correspondingly difficult. If someone pushes the door by applying a force at a point very close to hinges, it will be very difficult to open/close the door whatever be the magnitude of the force applied. If  $\mathbf{r}$  is the radius vector from the axis of rotation to the point where the force is applied, we may say that the rotational effect on the door depends upon the magnitude of  $\mathbf{r}$ . That is why a carpenter fixes the door handle near the edge of the door so that when we open or close the door using this handle, turning off the door becomes

easier. Although the example of opening/closing a door is used here to explain the rotational effect, this is true of all rotating bodies.

### **Inference II: Rotational effect is proportional to the magnitude of the radius vector**

Now try opening the door by keeping both the applied force and the distance constant, varying only the angle, say  $\theta$ , between radius vector  $\mathbf{r}$  and force  $\mathbf{F}$  (Fig. 7.3.1). You will find that the opening/closing of the door is easiest when the angle  $\theta$  is  $90^\circ$ . When the angle is gradually reduced, the rotation of the door becomes correspondingly difficult. When  $\theta = 0^\circ$  or  $180^\circ$ , the force is parallel to the door surface, rotation of the door is not possible. At all other angles, the rotational effect is proportional to  $\sin\theta$ .

### **Inference III: Rotational effect is proportional to $\sin\theta$**

With these inferences we introduce the idea of moment of force, also called torque.

#### **7.3.3 Moment of a force and torque**

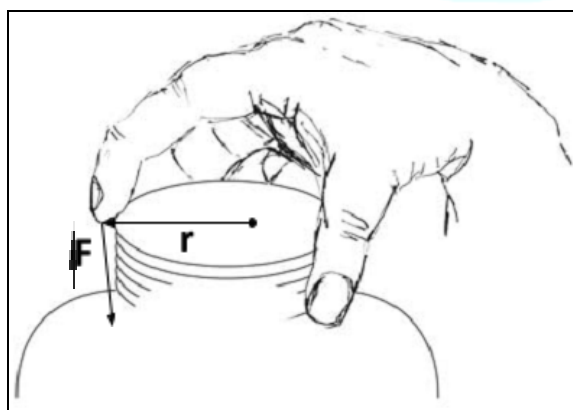
From all the above inferences, we find that we need to consider a physical quantity equal to  $rF\sin\theta$ , called **torque ( $\tau$ )** for studying rotational motion. The torque is also called the **moment of force**.

We can now state an equation for the magnitude of torque  $\tau$

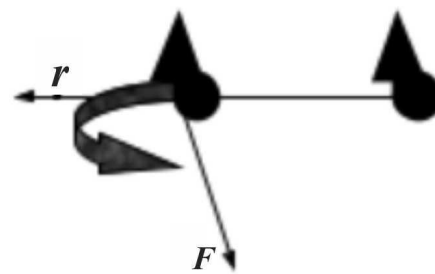
$$\tau = rF\sin\theta$$

In vector form, the equation would be

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$



(a)

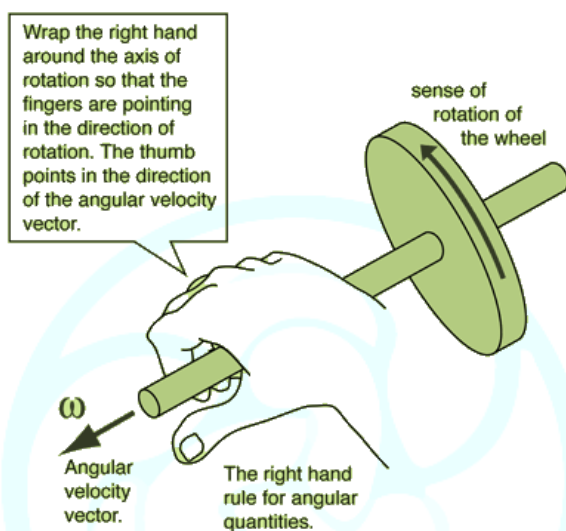


(b)

**Fig. 7.3.2: Opening lid of a can**

Note that the torque  $\tau$  is neither in the direction of  $r$  nor in the direction of  $F$ . It is perpendicular to both. Its direction is given by the right hand rule. If you rotate the vector  $r$  towards the vector  $F$  with your right hand through the smaller angle between them, the direction of motion of the lid will give you the direction of the torque (Fig.7.3.2(a)).

Another simple rule may also be used to give the direction of  $\omega$ . If one closes the right hand fist wherein curled fingers show the directions of rotation of a body, then the direction of the extended thumb denotes the direction of  $\omega$ . This is called the **right hand thumb rule**.



**Fig. 7.3.3 :** Right hand thumb rule showing the relation between angular velocity and axis of rotation

If the speed is increasing, the angular acceleration  $\alpha$  and the torque  $\tau$  are in the same direction as  $\omega$ . If the speed is decreasing, the angular acceleration and the torque are opposite to the direction of the angular velocity.

A similar action can be seen while opening the cork of a bottle by applying rotation about the axis of the bottle.



## Let Us Reflect

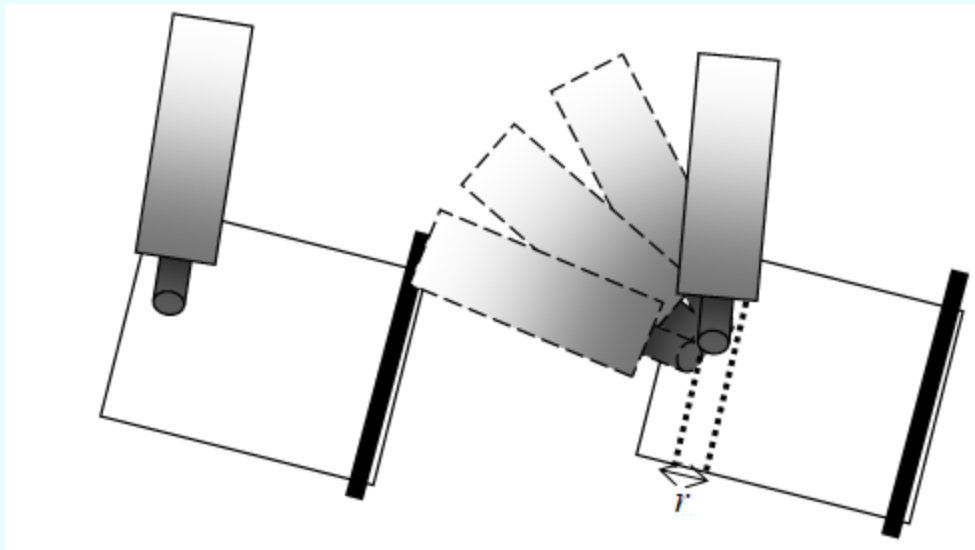
**Q.7.3.1** Two vectors **A** and **B** lie in the x-y plane making an angle  $\theta$  (less than  $180^\circ$ ) with each other. Using the right hand screw rule, find the direction of vector **C** equal to  $\mathbf{A} \times \mathbf{B}$ . What happens to the direction of vector **C** if the (i) direction of **A** or **B** is reversed, (ii) the direction of both is reversed.



## Answers

**A.7.3.1** According to the right hand screw rule the direction of vector  $\mathbf{C}$  would be perpendicular to the plane  $x$ - $y$  along the  $z$ -direction. (i) If the direction of one of the vectors is reversed, the direction of  $\mathbf{C}$  is also reversed, it is now in the direction of  $-z$ . (ii) If the direction of both the vectors is reversed,  $\mathbf{C}$  would again be in the  $z$ -direction.

### Activity 7.3.1-Pulling out a sheet of paper from beneath a water bottle



**Fig.7.3.4:** *Keep a PET bottle turned upside down on a paper laid on a table*

We take a bottle with a cap which has a plane surface. We know that the bottle is likely to topple about its narrow end when a torque is applied about the point of contact between the bottle and the table. Your activity is to pull out the paper without toppling the bottle with the help of a common writing pencil

(commonly known as lead pencil) without toppling the bottle. You are not expected to use any other means to hold or support the bottle.

It is possible to do so by applying the knowledge of torque and using some experimental skills. Keep the pencil parallel to the breadth of the paper as shown in Fig.7.3.4 and gradually roll the paper around the pencil skillfully. Rolling of the paper should be done so softly that it does not disturb the position of the bottle.

When you have rolled the paper sufficiently, the pencil is close to the bottle. Now roll very gently. One may expect the bottle to fall as soon as the pencil touches the bottle. Surprisingly it does not. Bottle, in fact, moves steadily along the pencil if you are rolling softly. If you can continue doing so, it may be possible to roll out the entire paper beneath the bottle.

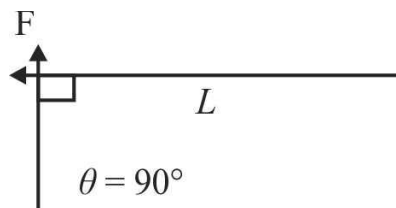
Now let us understand the physics in this activity.

We know that  $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$ , For magnitude of torque that acts on bottle is  $rF\sin\theta$ .  $\theta$  is the angle between vectors  $\boldsymbol{r}$  and  $\boldsymbol{F}$ . In our case, force  $\boldsymbol{F}$  is due to the pencil that pushes the bottle and  $\boldsymbol{r}$  is the distance between the pencil and the edge near it. Line touching the front edge of the bottle cap will act as the axis of rotation. Force  $\boldsymbol{F}$  is small as you are rolling gently as well as more importantly distance  $\boldsymbol{r}$  is small. Therefore, the torque acting on the bottle is too small to topple the bottle. However, the force acting due to rolling may push the bottle gradually out of paper and paper can be rolled out.

### 7.3.4 Lever arm of torque and angular momentum

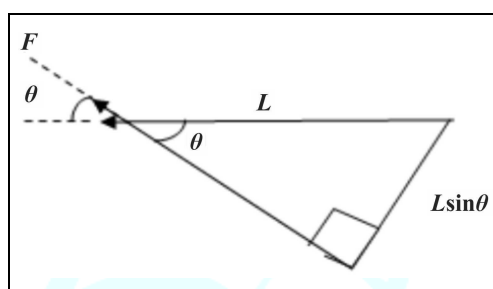
Perpendicular distance between the axis of rotation and line along which the force is applied is called the lever arm of the torque. If  $L$  is the shortest distance between the two, lever arm has magnitude of  $L\sin\theta$ , in general. Consider the following cases for clarity of this concept.

**Case I:** If force is perpendicular to radius vector, angle  $\theta = 90^\circ$ . Lever arm of the torque has length  $L$  (Fig. 7.3.5). It is the maximum value that it can have in a given system.



**Fig.7.3.5:** *Lever arm of torque*

**Case II:** Force makes an arbitrary angle  $\theta$  (Fig.7.3.6) with radius vector. Lever arm of the torque is the perpendicular distance between radius vector and the direction of force. It is equal to  $L \sin \theta$ .



**Fig. 7.3.6:** *Lever arm of torque at angle  $\theta$*

**Case III:** Force and the radius vectors are in the same or opposite directions. In these cases  $\theta = 0^\circ$  or  $180^\circ$ . Therefore, the lever arm  $L \sin 0^\circ = L \sin 180^\circ = 0$ . It is the minimum value of the lever arm for any system.

Similar to the correspondence between the force and the torque, let us look into the correspondence between the linear momentum and the angular momentum. If a point mass is moving around in a circular path, its angular momentum may be defined as the cross product of the radius vector  $\mathbf{r}$  of the particle with its linear momentum i.e.  $= m\mathbf{v}$ . Radius vector is measured from the axis of rotation. The angular momentum is given by

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = \mathbf{r} \times \mathbf{p}$$

The angular momentum has a magnitude of  $mvr$  and its direction is perpendicular to both  $\mathbf{r}$  and  $\mathbf{p}$ . It is determined by the right hand rule for vector products. The expression of the angular momentum suggests an alternative name for it. Angular momentum is also referred to as moment of momentum. This is as if we are taking the moment of the linear momentum  $m\mathbf{v}$  about the axis of rotation. So, when we say the angular momentum, we multiply the linear momentum  $m\mathbf{v}$  by  $\mathbf{r}$ . We may call lever arm as load arm or effort arm

depending upon whether force is load or effort. In equilibrium conditions, the moment of the load (a force) about the fixed point P, the fulcrum, is equal to the moment of the effort (another force) about the same fulcrum. The jack used for lifting heavy loads is a familiar example. When a metal bolt is opened by a spanner, the length of the spanner is the equivalent lever-arm, greater its value, more effective is the process of unscrewing.

### 7.3.5 Similarities between physical parameters related to linear and rotational motion

Rotational motion and linear motion can be studied considering the similarities between physical parameters related to the two types of motions. We tabulate such corresponding quantities in the Table 7.3.1 given below. You may note in the Table that a term for radius vector  $\mathbf{r}$  appears invariably in equations of almost all physical quantities of rotational motion,.

**Table 7.3.1** *Corresponding quantities of linear and angular motion*

| Linear Motion                                 | Rotational Motion                                                |
|-----------------------------------------------|------------------------------------------------------------------|
| Linear Displacement ( $r$ )                   | Angular Displacement ( $\theta$ )                                |
| Linear Velocity ( $v$ ) = $\frac{dr}{dt}$     | Angular Velocity ( $\omega$ ) = $\frac{d\theta}{dt}$             |
| Linear Acceleration ( $a$ ) = $\frac{dv}{dt}$ | Angular Acceleration ( $\alpha$ ) = $\frac{d\omega}{dt}$         |
| Force ( $F$ )                                 | Torque ( $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ )    |
| Inertia ( $m$ )                               | Moment of Inertia ( $I = \sum mr^2$ )                            |
| Linear Momentum ( $p = mv$ )                  | Angular Momentum ( $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ ) |

Vector relation between linear and angular velocity:  $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

Vector relation between linear and angular acceleration:  $\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r}$

We have learnt that force by itself cannot explain rotational motion; an equivalent quantity called moment of force (torque) is required for it.

Torque depends upon the radius vector  $\mathbf{r}$  from the axis of rotation to the point of application of force. Similarly, for other corresponding quantities in rotational motion have radius vector  $\mathbf{r}$  in their expression. Inertia in linear

motion is represented by mass (a scalar quantity), while inertia in rotational motion is represented by the corresponding quantity, Moment of Inertia. It is the product of the mass of the particle and the square of the distance between the particle and the axis of rotation. All physical quantities shown in Table 7.3.1 change when the axis of rotation changes. Inertia in linear motion is constant (if  $m$  is constant), whereas moment of inertia in rotational motion can change when the axis of rotation changes although the mass remains constant. The moment of inertia of some regular shaped bodies about specific axes are given in Table 7.3.2.

**Table 7.3.2: Moment of Inertia of some Regular Bodies**

| Z   | Body                           | Axis                               | Figure                                                                               | I         |
|-----|--------------------------------|------------------------------------|--------------------------------------------------------------------------------------|-----------|
| (1) | Thin circular ring, radius $R$ | Perpendicular to plane, at centre  |    | $MR^2$    |
| (2) | Thin circular ring, radius $R$ | Diameter                           |    | $MR^2/2$  |
| (3) | Thin rod, length $L$           | Perpendicular to rod, at mid point |   | $ML^2/12$ |
| (4) | Circular disc, radius $R$      | Perpendicular to disc at centre    |  | $MR^2/2$  |
| (5) | Circular disc, radius $R$      | Diameter                           |  | $MR^2/4$  |
| (6) | Hollow cylinder, radius $R$    | Axis of cylinder                   |  | $MR^2$    |
| (7) | Solid cylinder, radius $R$     | Axis of cylinder                   |  | $MR^2/2$  |
| (8) | Solid sphere, radius $R$       | Diameter                           |  | $2MR^2/5$ |

Directions of vector quantities in the table of rotational motion versus linear motion, under the rotational motion column are found by right hand rule.

Let us discuss the case of a rotating wheel to understand the linear and angular momentum.

1. If one wishes to track the direction of motion of a point on a rotating wheel, it may be inconvenient as the direction of linear velocity changes at every moment. However, the angular velocity, being a vector quantity, should have a unique direction. What could be this direction? The direction of the angular velocity is taken as the direction of the axis of rotation. The magnitude of the angular velocity vector is the speed of rotation of the wheel. However, the angular velocity vector can change if the wheel is precessing.
2. For a particle moving in a circular path, the direction of its linear velocity is continuously changing, leading to a changing linear momentum. For a particle moving uniformly in a circular path, the angular momentum vector, however, is a fixed quantity (both in magnitude and direction) if the axis does not undergo any precessional motion.



### Let Us Reflect

**Q.7.3.2** A spherical block of wax has radius  $R$ . What would be its moment of inertia about an axis passing through its centre. If this block of wax is converted into a cylinder of length  $L$ . What would be the moment of inertia of this cylinder along an axis passing through its centre and perpendicular to its axis. Would the moment of inertia be the same in both cases? Give reason for your answer.



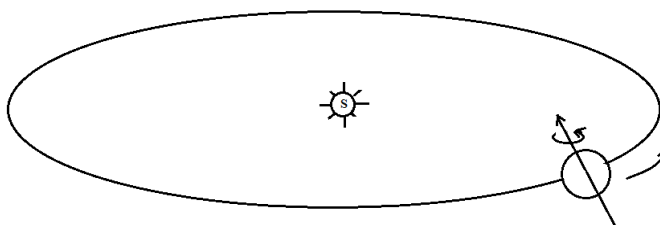
### Answers

**A.7.3.2** The moment of inertia in two cases would not be the same. The reason is that the distribution of mass around the axis of rotation has changed.

Moment of inertia of the sphere =  $\frac{1}{2}MR^2$

Moment of inertia of the cylinder about the axis passing through its centre and perpendicular to its length =  $\frac{1}{3}ML^2$ .

### 7.3.6 Rotation and spin



**Fig. 7.3.7:** *Earth spins around its axis of rotation and also revolves around the Sun in a nearly circular orbit*

A very common statement that all of us are familiar with is ‘the earth spins about its own axis causing day/night and revolves around the sun causing different seasons’. A spinner bowling to a batter in the game of cricket applies a torque on the ball in such a way that the ball spins about one of its diameters. This axis of spinning passes through the ball and we say the ball is spinning. On the other hand, if the ball moves in a circular orbit about another body, then its motion is called rotation (Fig. 7.3.7).

Both for spin or rotation we need to apply a torque about the suitable axis. All the principles of rotational motion are applicable in both the cases. For a point mass, however, we cannot have any spin as theoretically a point mass does not have any spread about any axis passing through it. But, it can still rotate about an axis.

The ceiling fan suspended from the ceiling of a room actually spins about the vertical axis passing through its centre, and the axis is perpendicular to the plane of spin. On the other hand, in the case of a cabin fan fixed on a wall and facing towards the floor, the axis about which the fan spins is not vertical but it is perpendicular to the plane of rotation. The table and pedestal fans usually spin about a horizontal axis. However, the whole system of blades with metal cage may itself be rotating about an axis of rotation oscillating in a plane, for distributing air in different directions. So, a table fan or a pedestal fan has both spinning and rotational motion about two different axes.

### 7.3.7 Laws of rotational motion

The law governing the rotation is analogous to the second law of Newton:

**The rate of change of angular momentum is directly proportional to the applied torque and it takes place in the same sense in which the torque is applied, i.e.,  $\tau = \frac{dL}{dt}$ .**

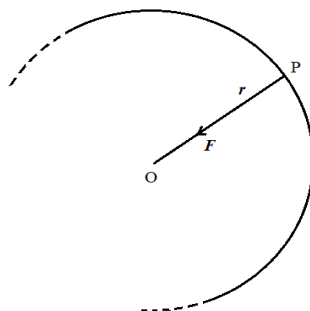
### Some discussions on angular momentum and its conservation

Let us consider the case of a central orbit, that is the motion of a body under a force directed towards the centre of the orbit as an illustrative example.

A central force is given by,

$$\mathbf{F} = f(r) \hat{\mathbf{r}} \quad (7.3.1)$$

where  $\hat{\mathbf{r}}$  is the unit vector along the direction of the radius vector,  $\mathbf{r}$ . The trajectory is shown schematically in Fig.7.3.8. The force  $\mathbf{F}$  is directed along PO, that is towards the centre; in other words we are considering an attractive force.



**Fig.7.3.8:** A body moving under the influence of a central force

The torque is given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

$$\begin{aligned}
 &= \hat{\mathbf{r}} \times \mathbf{f}(\mathbf{r}) \hat{\mathbf{r}} = r \hat{\mathbf{r}} \times f(r) \hat{\mathbf{r}} \\
 &= 0 \quad (\because \hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0)
 \end{aligned}$$

$$\therefore \boldsymbol{\tau} = 0 \text{ and hence } \frac{d\mathbf{L}}{dt} = 0 \quad (7.3.2)$$

$$\text{Or } \mathbf{L} = \text{a constant vector, angular momentum of the body.} \quad (7.3.3)$$

Eq(7.3.2) has two important consequences.

### Consequence I

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} \quad (7.3.4)$$

where  $\mathbf{v}$  is the velocity of the body of mass  $m$  moving under the influence of a central force.

Now, from Eq(7.3.4)

$$\begin{aligned}
 \mathbf{r} \cdot \mathbf{L} &= \mathbf{r} \cdot (\mathbf{r} \times \mathbf{p}) = \mathbf{r} \cdot (m \mathbf{r} \times \mathbf{v}) \\
 &= m(\mathbf{r} \cdot (\mathbf{r} \times \mathbf{v}))
 \end{aligned}$$

$(\mathbf{r} \times \mathbf{v})$  is a vector perpendicular to  $\mathbf{r}$ .

$$\therefore \mathbf{r} \cdot (\mathbf{r} \times \mathbf{v}) = 0$$

$$\therefore \mathbf{r} \cdot \mathbf{L} = 0$$

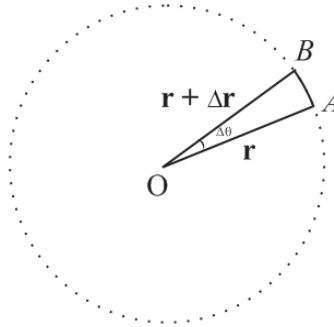
In other words,  $\mathbf{L}$  is perpendicular to  $\mathbf{r}$ , that is the radius vector  $\mathbf{r}$  is always perpendicular to a constant vector in orbital motion. So, the radius vector always lies in a plane perpendicular to the constant vector  $\mathbf{L}$ . So the motion of a body/ particle under the influence of a central force is confined to a plane.

It means that the motion of a planet about the sun takes place in a plane. So does the motion of an electron orbiting the nucleus in the hydrogen atom.

Most importantly, the angular momentum of a body moving under the influence of a central force remains conserved. In general, if the resultant torque acting on a body is zero, its angular momentum is conserved. This is the principle of conservation of angular momentum.

## Consequence II

Let us again consider the trajectory of a particle under the influence of a central force. We now know that the trajectory is a planar orbit. O is the origin of the force. The particle moves from A to B in a time  $\Delta t$  by an infinitesimal distance. So, the area swept out by the radius vector in time  $\Delta t$  is the area of triangle OAB. (Fig.7.3.9) Since the arc AB is infinitesimally small, we may consider the area swept out as the area of  $\Delta OAB = \Delta A$ , say.



**Fig. 7.3.9:** A particle sweeping out an area AOB, by describing an infinitesimal angle  $\Delta \theta$

$$\begin{aligned}\Delta A &= \frac{1}{2} |\mathbf{OA} \times \mathbf{OB}| = \frac{1}{2} |\mathbf{r} \times (\mathbf{r} + \Delta \mathbf{r})| = \frac{1}{2} |(\mathbf{r} \times \mathbf{r}) + (\mathbf{r} \times \Delta \mathbf{r})| \\ &= \frac{1}{2} |\mathbf{r} \times \Delta \mathbf{r}| \quad (\because \mathbf{r} \times \mathbf{r} = 0) \\ \therefore \frac{\Delta A}{\Delta t} &= \frac{1}{2\Delta t} |\mathbf{r} \times \Delta \mathbf{r}|\end{aligned}$$

Since the division by  $\Delta t$  is independent of the cross product operation, we may write,

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} \left| \mathbf{r} \times \frac{\Delta \mathbf{r}}{\Delta t} \right|$$

The areal velocity that is the rate at which the area is swept out by the radius vector can be expressed as  $\frac{dA}{dt}$ .

$$\begin{aligned}\text{And, } \frac{dA}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta A}{\Delta t} = \frac{1}{2} \lim_{\Delta t \rightarrow 0} \left| \mathbf{r} \times \frac{\Delta \mathbf{r}}{\Delta t} \right| \\ &= \frac{1}{2} \left| \mathbf{r} \times \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} \right|, \text{ as the operation of taking limit as } \Delta t \rightarrow 0 \text{ is independent of the} \\ &\text{operation of cross product.}\end{aligned}$$

$$\therefore \frac{dA}{dt} = \frac{1}{2} \left| \mathbf{r} \times \frac{d\mathbf{r}}{dt} \right| = \frac{1}{2} |\mathbf{r} \times \mathbf{v}| \quad (7.3.5)$$

But from Eqs(7.3.3) and (7.3.4), we get

$$m(\mathbf{r} \times \mathbf{v}) = a \text{ constant}$$

$$\text{Or, } \mathbf{r} \times \mathbf{v} = a \text{ constant}$$

From Eq(7.3.5),

$$\frac{dA}{dt} = a \text{ constant} \quad (7.3.6)$$

$$\text{Also, as } L = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v},$$

$$\text{we have } \frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

That is, the radius vector sweeps out equal areas in equal intervals of time, which is Kepler's second law.

It is interesting to note that the conservation of the areal velocity does not depend on the form of force  $f(r)$ . It is only required that the force should be central. As a matter of fact, the expression of  $f(r)$  for the Sun and a planet system is  $\frac{GMm}{r^2}$ , where the symbols have their usual meaning, but had the form not even been inverse square, Kepler's second law would still have been valid.

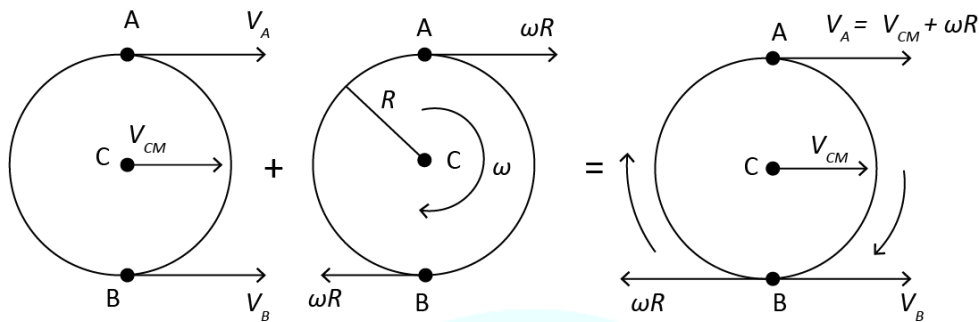
Angular momentum plays a crucial role in deciding the orbits of an electron in the hydrogen atom through the Bohr model. In the journey of modification from Rutherford to Bohr model, the latter model had to depend on a conserved quantity. For a central orbit, angular momentum is indeed a conserved quantity. He chose angular momentum to postulate the quantum condition as a criteria for permitted or allowed orbits, as:

$$mvr = \frac{nh}{2\pi}, \text{ n is an integer} \quad (7.3.7)$$

and thereby the orbits got quantized.

### 7.3.8 Condition for rolling without slipping

Rolling motion is a general motion which is the combination of translatory and rotational motion. Fig. 7.3.10 (a) shows pure translational motion, Fig. 7.3.10 (b) shows pure rotational motion and Fig. 7.3.10 (c) shows the rolling motion.



**Fig. 7.3.10:** (a) Pure translational motion, (b) Pure rotational motion and (c) the rolling motion

Consider three points A (topmost), B (in the contact of ground) and C (at Centre of mass) in Fig. 7.3.10

Considering only magnitudes:

For translatory motion

$$v_A = v_B = v_{CM} \quad (7.3.8)$$

For rotational motion, if angular velocity is  $\omega$  and radius is  $R$ , velocity of a point on the circumference will be

$$= \omega R \text{ (tangential to the circumference)} \quad (7.3.9)$$

For rolling motion,

$$v_{A(\text{NET})} = v_A + \omega R$$

$$v_{B(\text{NET})} = v_B - \omega R$$

If now the body rolls and does not slip then point B is at rest at the moment of contact with the surface.

$$\therefore v_{B(\text{NET})} = 0$$

$$\text{or } v_B - \omega R = 0$$

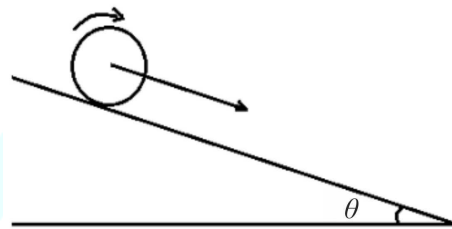
$$\text{or } v_B = \omega R$$

From Eq(7.3.8)

$$v_{CM} = \omega R \quad (7.3.10)$$

i.e., for rolling without slipping the centre of mass of the body should exhibit translatory motion with velocity  $\omega R$ .

### 7.3.8.1 A body rolling without slipping down on inclined plane



**Fig.7.3.11 :** A body rolling down an inclined plane of inclination  $\theta$

For a body rolling down an inclined plane of inclination  $\theta$  (Fig.7.3.11) with the horizon, the change in kinetic energy (KE) for coming down a distance  $s$  down the incline is  $smg\sin\theta$ . It is equal to the sum of gain in KE.

KE = KE of translation + KE of rotational

$$= \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I\omega^2$$

$I$  can be expressed as,  $I = mK^2$ , where  $K$  is called the radius of gyration, and

$$\omega = \frac{v_{CM}}{R}$$

from Eq(7.3.10)

$$\therefore \text{Gain in KE} = \frac{1}{2} \left( 1 + \frac{K^2}{R^2} \right) m v_{CM}^2 = mg \sin \theta$$

$$\text{Or } v_{CM}^2 = \frac{2sg \sin \theta}{1 + \frac{K^2}{R^2}}$$

$$\therefore \text{The corresponding acceleration is given by } a = \frac{v^2}{2s} = \frac{g \sin \theta}{1 + \frac{K^2}{R^2}}$$

By knowing the values of the radius of gyration of several objects of regular shape, we may determine the velocity and acceleration of bodies rolling without slipping. For example, for a hollow cylinder (HC) and solid cylinder (SC) of the same radius  $R$ ,  $K^2 = R^2$  and  $\frac{K^2}{2}$  respectively. So, the acceleration of a hollow cylinder and a solid cylinder are given by,

$$a_{HC} = \frac{g \sin \theta}{1+1} = \frac{g \sin \theta}{2}$$

$$a_{SC} = \frac{g \sin \theta}{1+\frac{1}{2}} = \frac{2}{3} g \sin \theta$$

**In general, the radius of gyration is not the distance of the centre of mass from the axis of rotation.**

This is a common misconception that the radius of gyration is the distance of the centre of mass from the axis of rotation. In fact, it is the distance of the point where the whole mass is assumed to be concentrated so that the moment of inertia (MI) is the same as with the actual distribution of mass. That can be seen from the example of rod.

MI of a uniform rod about its perpendicular bisector is

$$I = \frac{1}{12} m l^2 = m K^2$$

$$K = \frac{l}{\sqrt{12}}$$

$K$  = radius of gyration

MI of the rod about an axis through an end is

$$I = mK^2 = \frac{1}{3} ml^2$$

$$K = \frac{l}{\sqrt{3}} l$$

Hence, it is clear that the radius of gyration is not the distance of the centre of mass from the axis of rotation. Radius of gyration depends on the axis of rotation and for the same body, the radius of gyration may be different for different axes of rotation but the centre of mass is the same for all axes of rotation.

For a ring rotating about its axis the radius of gyration is equal to the radius of the ring when the centre of mass lies on the axis itself.

if  $K = R$ , then the distance of the centre of mass from the axis of rotation is zero.

### 7.3.9 Equilibrium and its types

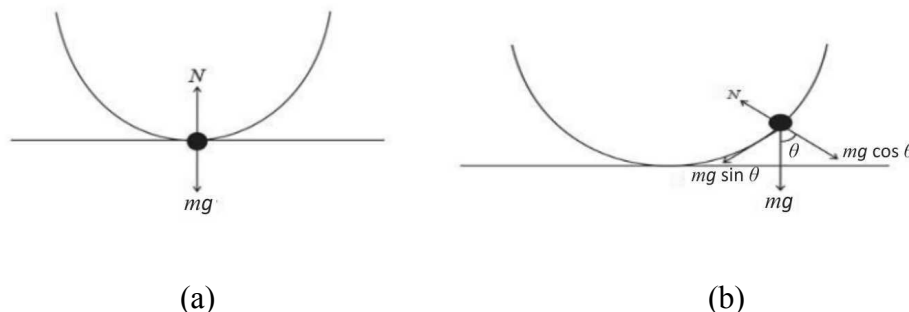
A system is said to be in equilibrium if the net force and the net torque on it are zero, i.e.,  $\mathbf{F}_{net} = 0$  and

$$\tau_{net} = 0.$$

#### 7.3.9.1 Types of equilibrium of moving bodies

**(1) Stable equilibrium:** For stable equilibrium  $\mathbf{F}_{net} = 0$ ,  $\tau_{net} = 0$  and potential energy of system  $U$  is minimum. If the system is displaced from the position of stable equilibrium, then the system tends to return back to the stable equilibrium at its own.

Example: A marble in a hemispherical bowl at its lowermost point.

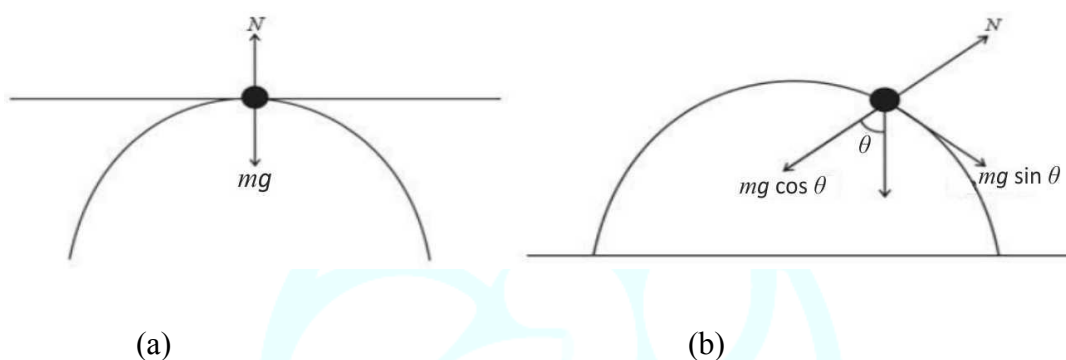


**Fig. 7.3.12:** *Stable equilibrium*

In Fig. 7.3.12, if the marble is displaced from the lowermost point and released, then it returns back to the lowermost point on its own. Also,  $N \cos \theta$  is equal and opposite to  $mg$ , and  $N \sin \theta$  provides the restoring force. This leads to oscillations of the marble about the equilibrium position.

**(2) Unstable equilibrium:** For unstable equilibrium,  $F_{net} = 0$  and the potential energy of the system,  $U$  is maximum. If the system is displaced from the position of unstable equilibrium, then the system cannot return back to the equilibrium state on its own.

**Example:** A marble at the topmost point on the inverted hemispherical bowl.



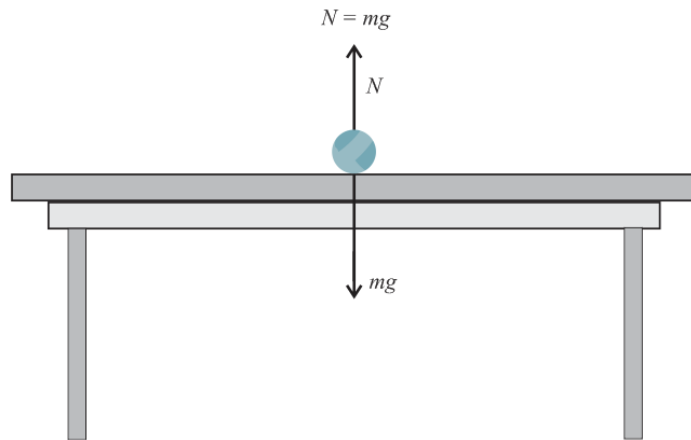
**Fig. 7.3.13:** *Unstable equilibrium*

In Fig.7.3.13, if the marble is displaced from the topmost point, it rolls down and never comes back to its original position at the topmost point on its own. Here,  $N \sin \theta$  acts as a destabilising force and the marble does not move towards the equilibrium position and oscillation does not take place.

### Neutral equilibrium

A system is said to be in a state of neutral equilibrium if  $F_{net}$  and potential energy of the system are constant. If a system is displaced from neutral equilibrium, it will change its position but there is no accompanying change in the potential energy of the system.

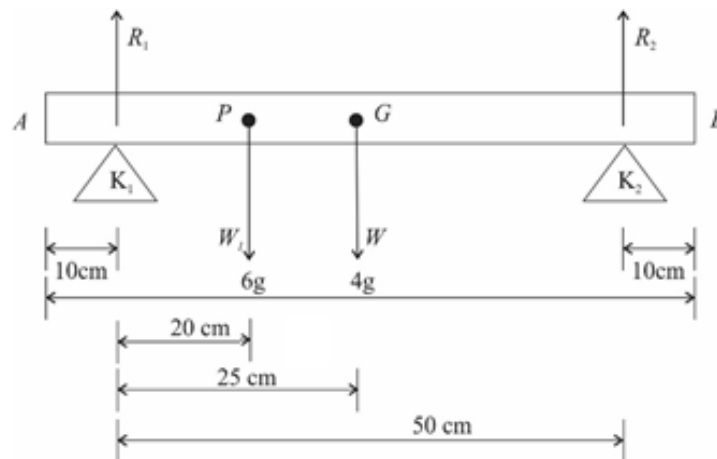
**Example:** When a marble placed on a plane level surface is displaced, it does not return to its initial position, since its potential energy does not change (Fig. 7.3.14).



**Fig. 7.3.14:** Neutral equilibrium

### 7.3.9.2 Position of the axis of rotation and equilibrium

Where should the axis of rotation be taken to solve problems of equilibrium? Let us discuss.



**Fig. 7.3.15** Position of the axis of rotation and equilibrium

If the body is rotating, we do not have any choice. The axis about which the body is rotating is to be taken and all moments of forces are to be estimated about this axis only. But, what if the body is in equilibrium or at rest? The moment of force is calculated around the centre of mass (or centre of gravity). It helps remove the misconception that the axis should always be coincident with the axis of rotation. It is not so, for a body at rest.

We can take the axis of rotation anywhere as per our convenience, so that calculation of moment of force is easy.

Consider a 70 cm long rod (Fig.7.3.15). Knife edges are 10 m away from each end. A 6 kg load is suspended at 30 cm from one end.

We take the axis of rotation at 30 cm from the end A.

For translatory equilibrium,

$$R_1 + R_2 = (6 \text{ kg} + 4 \text{ kg})g$$

$$\text{Or } R_1 + R_2 = 98 \text{ N} \quad (7.3.11)$$

For rotational equilibrium, let us consider point G as the point of reference. Here,

Clockwise torque = Anticlockwise torque

$$\text{Hence, } R_1 \times (K_1G) = W_1 \times (PG) + R_2 \times (K_2G)$$

$$R_1 \times (25 \text{ cm}) = (6 \text{ kg})g \times (5 \text{ cm}) + R_2 \times (25 \text{ cm})$$

$$\text{Or } (25 \text{ cm}) (R_1 - R_2) = (30 \text{ kg})g \text{ cm}$$

$$\text{Or } R_1 - R_2 = (1.2 \text{ kg})g$$

$$= 1.2 \times 9.8 \text{ N}$$

$$\text{i.e. } R_1 - R_2 = 11.76 \text{ N}$$

Solving this equation with that of  $(R_1 + R_2)$ , we get  $R_1 = 54.88 \text{ N}$ ,  $R_2 = 43.12 \text{ N}$ .

If the axis of rotation is chosen at a different point, the values of  $R_1$  and  $R_2$

will change.



## Let Us Reflect

**Q.7.3.3** A tight flexible rope of negligible weight is wound round a solid cylinder of mass  $M$  and radius  $R$  (Fig.7.3.16). The rope carries a mass  $m$  at its free end which is at rest at a height  $h$  above the floor. Assuming friction between the rope and the surface of the cylinder to be absent, which keeps rotating about its own axis, find the tension in the rope and the acceleration of the mass  $m$  after its release.

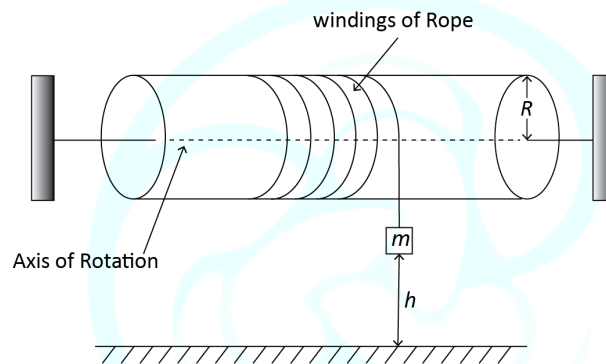


Fig.7.3.16



## Answers

**A.7.3.3** The potential energy of the mass  $m = mgh$

When the mass gets released, the potential energy of the mass is converted into the kinetic energy of its translation and the kinetic energy of rotation of the cylinder.

We write the energy relation as,

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \quad (7.3.12)$$

We know that,

$$I = \frac{1}{2}MR^2 \text{ and } \omega = \frac{v}{R}$$

$$\therefore mgh = \frac{1}{2}m\omega^2 R^2 + \frac{1}{2} \cdot \frac{1}{2}MR^2\omega^2$$

$$\text{or, } 4mgh = \omega^2 R^2 (2m + M)$$

$$\text{or } \omega^2 = \frac{4mgh}{R^2(2m+M)}$$

$$\text{Now, } v^2 = \omega^2 R^2 = \frac{4mgh}{2m+M} \quad (7.3.13)$$

Let the acceleration of the mass be  $a$ .

$$\text{As it starts from rest, } v^2 = 2ah \quad (7.3.14)$$

Combining Eqs(7.3.13) and (7.3.14), we get

$$a = \frac{2mg}{2m+M}$$

If the tension in the rope is  $T$ , we have from the free body diagram,

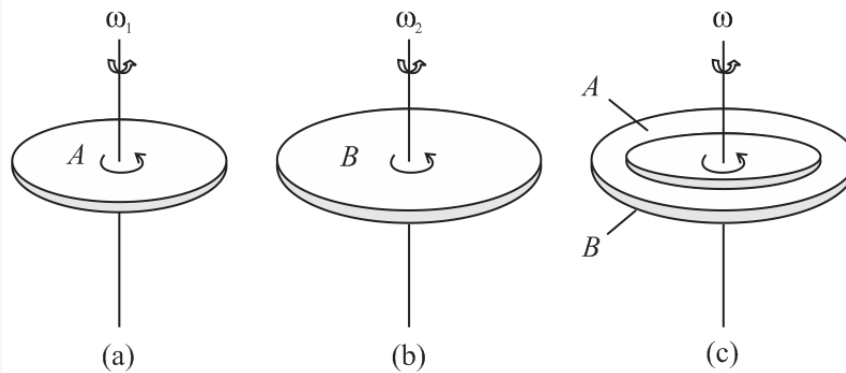
$$mg - T = ma$$

$$\text{Or, } T = m(g - a)$$

$$= m\left(g - \frac{2mg}{2m+M}\right) = \frac{Mmg}{2m+M}.$$

## 7.3.10 A few illustrative problems on dynamics of rotational motion

**Example 7.3.1** Refer to the Fig. 7.3.17(a), (b), (c), given below-



**Fig. 7.3.17:** Rotational Motion of two discs individually ((a), (b) and combined (c))

In Fig. 7.3.17 A and B are two discs rotating about their axes (passing through their centre and perpendicular to their planes) with angular velocities  $\omega_1$  and  $\omega_2$  respectively. Their moments of inertia about their axes are  $I_1$  and  $I_2$  respectively. They are brought in contact as shown in Fig 7.3.17 (c), with the discs one above the other and the axis becoming coincident. Find the angular velocity of the combined system.

Is the kinetic energy of the combined system the same as the sum of individual kinetic energies of the two discs? If not, obtain an expression for the change in kinetic energy.

**Solution:**

The total initial angular momentum of the system

$$= I_1 \omega_1 + I_2 \omega_2 \quad (7.3.15)$$

The moment of inertia of the combined system  $= I_1 + I_2$ . Let the angular velocity of the combined system be  $\omega$ . Then the angular momentum of the combined system  $= (I_1 + I_2)\omega$ . Since there is no external torque, the angular momentum will be conserved.

$$\therefore (I_1 + I_2)\omega = I_1 \omega_1 + I_2 \omega_2$$

$$\text{Or, } \omega = \frac{I_1 \omega_1 + I_2 \omega_2}{I_1 + I_2} \quad (7.3.16)$$

We know that moment of inertia is the rotational analogue of mass and angular velocity is the analogue of linear velocity. So, the kinetic energy of a rotating system is  $\frac{1}{2}I\omega^2$ .

Now, the sum of the individual kinetic energies of the two discs is,

$$K_1 = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2$$

And the kinetic energy of the combined system is  $K_2 = \frac{1}{2}I\omega^2$

$$\therefore K_2 = \frac{1}{2}(I_1 + I_2) \left( \frac{I_1 \omega_1 + I_2 \omega_2}{I_1 + I_2} \right)^2$$

The difference in KE =  $K_1 - K_2$

$$\begin{aligned} &= \frac{1}{2}I_1 \omega_1^2 + \frac{1}{2}I_2 \omega_2^2 - \frac{1}{2}(I_1 + I_2) \left( \frac{I_1 \omega_1 + I_2 \omega_2}{I_1 + I_2} \right)^2 \\ \therefore K_1 - K_2 &= \frac{1}{2} \left\{ I_1 \omega_1^2 + I_2 \omega_2^2 - \frac{(I_1 \omega_1 + I_2 \omega_2)^2}{I_1 + I_2} \right\} \end{aligned} \quad (7.3.17)$$

Eqn. (7.3.14), is exactly similar to the expression  $K_1 - K_2 = \frac{1}{2} \left\{ m_1 u_1^2 + m_2 u_2^2 - \frac{(m_1 u_1 + m_2 u_2)^2}{m_1 + m_2} \right\}$  of the two-body collision problem wherein the two bodies combine into a single body after collision.

After the necessary calculations, as done in case of Example 7.3.9, we arrive at the result,

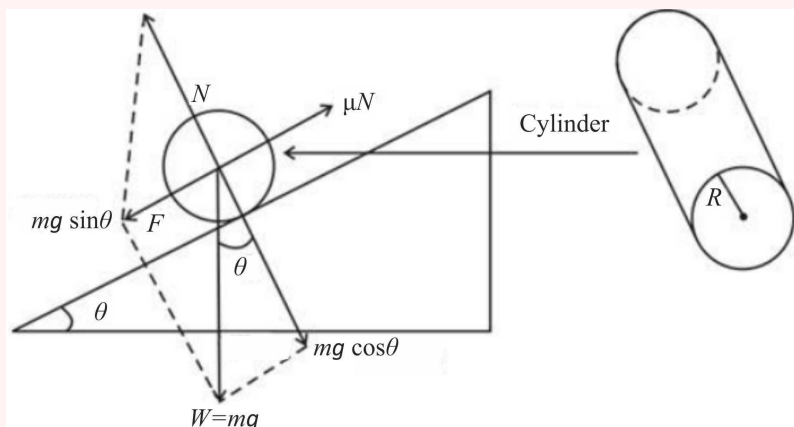
$$K_1 - K_2 = \frac{1}{2} \frac{I_1 I_2}{I_1 + I_2} (\omega_1 - \omega_2)^2 \quad (7.3.18)$$

which is similar to the expression  $\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (u_1 - u_2)^2$ , i.e. loss of KE in a two-body system after their collision.

In both the cases, linear and rotating, there occurs loss in kinetic energy. In the first case, the bodies come in contact, whereas in the second case, the two discs come in contact. In each case we get a combined system. In the process of becoming a combination and thereby attaining a common speed, be it linear or angular, there occurs a loss in energy. This loss is due to frictional forces. The loss in kinetic energy is a consequence of work done against the forces of friction. The loss is by and large dissipated in the form of heat.

**Example 7.3.2**

A cylinder rolls down an inclined plane of slope  $\theta$  (Fig. 7.3.18). Find the condition for the cylinder to slip if  $\mu$  is the coefficient of friction between the surface of the cylinder and the inclined plane.



**Fig. 7.3.18:** Free-body diagram of the cylinder rolling down an inclined plane

**Solution:**

The free-body diagram with all the relevant forces is self-explanatory. Now, let the linear acceleration of the cylinder down the plane be  $a$ . Considering forces along the inclined plane:

$$mg \sin \theta - \mu N = ma$$

Since the net force is zero in direction perpendicular to the inclined surface, we have

$$mg \cos \theta - N = 0. \text{ Therefore,}$$

$$m(g \sin \theta - \mu g \cos \theta) = ma$$

$$\text{or } a = g(\sin \theta - \mu \cos \theta) \quad (7.3.19)$$

The torque due to frictional force  $= \mu N R = \mu m g R \cos \theta$ , where  $R$  = the radius of the cylinder.

The moment of inertia of the cylinder about its axis  $= \frac{1}{2} m R^2 = I$  and the torque about the axis of the cylinder  $= I \alpha$ , where  $\alpha$  is the angular acceleration of the cylinder.

$$\therefore I \alpha = \mu m g R \cos \theta$$

$$\therefore \alpha = \frac{\mu m g R \cos \theta}{\frac{1}{2} m R^2} = \frac{2 \mu g \cos \theta}{R}$$

If the cylinder rolls without slipping, then its acceleration,

$$R \alpha = 2 \mu g \cos \theta \quad (7.3.20)$$

The cylinder slips if,

$$R\alpha' < a$$

From Eqs(7.3.19) and (7.3.20), we get  $2\mu g \cos \theta < (g \sin \theta - \mu g \cos \theta)$

$$\therefore 3\mu g \cos \theta < g \sin \theta$$

The required condition is  $\mu < \frac{1}{3}\tan \theta$ .

**Example 7.3.3** A particle of mass  $m$  moves along a curve defined by,  $\mathbf{r} = A \cos \omega t \hat{\mathbf{i}} + B \sin \omega t \hat{\mathbf{j}}$ . Find (a) the torque acting on the particle and (b) the angular momentum of the particle about the origin.

**Solution:**

$$\mathbf{r} = A \cos \omega t \hat{\mathbf{i}} + B \sin \omega t \hat{\mathbf{j}} \quad (7.3.21)$$

$$\therefore \frac{d\mathbf{r}}{dt} = \mathbf{v} = -\omega A \sin \omega t \hat{\mathbf{i}} + \omega B \cos \omega t \hat{\mathbf{j}}$$

$$\frac{d\mathbf{v}}{dt} = \mathbf{a} = -A\omega^2 \cos \omega t \hat{\mathbf{i}} - B\omega^2 \sin \omega t \hat{\mathbf{j}} \quad (7.3.22)$$

And, acceleration,

$$= -\omega^2 (A \cos \omega t \hat{\mathbf{i}} + B \sin \omega t \hat{\mathbf{j}})$$

$$\therefore \mathbf{a} = -\omega^2 \mathbf{r} \quad (7.3.23)$$

So, the force acting on the body =  $\mathbf{F} = m\mathbf{a}$

$$\mathbf{F} = -m\omega^2 \mathbf{r} \quad (7.3.24)$$

$$\text{Torque} = \mathbf{r} \times \mathbf{F}$$

$$= -\mathbf{r} \times m\omega^2 \mathbf{r}$$

$$= 0 \quad (\because \mathbf{r} \times \mathbf{r} = 0)$$

$$\text{Angular momentum } L = \mathbf{r} \times m\mathbf{v}$$

$$\therefore \mathbf{L} = m\omega \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos \omega t & b \sin \omega t & 0 \\ -a \sin \omega t & b \cos \omega t & 0 \end{vmatrix}$$

or

$$= m\omega (ab \cos^2 \omega t + ab \sin^2 \omega t) \hat{k}$$

$$= mab\omega (\cos^2 \omega t + \sin^2 \omega t) \hat{k} = mab\omega \hat{k}$$

$\mathbf{L}$  is a constant independent of  $t$ ,

$$\therefore \frac{d\mathbf{L}}{dt} = 0$$

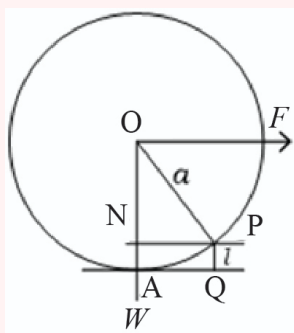
Hence, in the absence of a net external torque, the angular momentum of the system remains constant.

**Example 7.3.4** Fig.7.3.19 shows a heavy carriage of weight  $W$  and radius  $a$ . It is to be dragged over an obstacle of height  $l$ , as shown, by the application of a horizontal force  $F$  applied at the centre of the wheel.

Show that  $F$  must be slightly greater than  $\frac{N}{a-1} (2al - l^2)^{1/2}$ .

**Solution:** The carriage wheel with centre at  $O$ , and touching the ground at  $A$ , has a weight  $W$ . The obstacle  $PQ$  is of height  $l$ . In order to be dragged over the obstacle, the moment of the force  $F$  about  $P$  should be greater than the moment of the weight  $W$  about  $P$ . In other words,

$$F(ON) \geq W(PN) \quad (7.3.25)$$



**Fig. 7.3.19** : Torques are due to applied force  $F$  and weight  $W$  about the axis of rotation at point  $P$ .

In order to just overcome the obstacle  $F(ON)$  has to be slightly greater than  $W(PN)$

So we shall consider the ‘>’ sign in Eq(7.3.25) to read as ‘slightly greater’.

Now,  $ON = OA - AN = a - l$

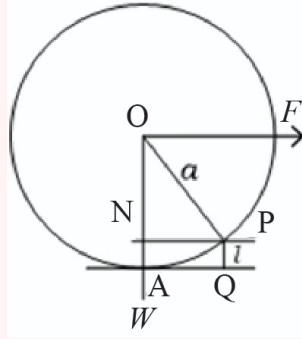
$$\begin{aligned}
 PN &= \sqrt{OP^2 - ON^2} \\
 &= \sqrt{a^2 - (a - l)^2} \\
 &= \sqrt{a^2 - a^2 + 2al - l^2} \\
 &= (2al - l^2)^{1/2} \\
 &= \sqrt{a^2 - (a - l)^2} \\
 &= \sqrt{a^2 - a^2 + 2al - l^2} \\
 &= (2al - l^2)^{1/2}
 \end{aligned}$$

∴ From Eq(7.3.25), we get

$$\begin{aligned}
 F(a - l) &\geq W(2al - l^2)^{1/2} \\
 \text{or } F &\geq \frac{W}{a - l}(2al - l^2)^{1/2}
 \end{aligned}$$

**Example 7.3.5** A particle of mass  $m$  is projected at time  $t = 0$  from a point P on the ground with a speed  $v_0$ , at an angle  $45^\circ$  to the horizontal. Find the magnitude and direction of the angular momentum of the particle about P at time  $t = v_0/g$ .

**Solution:** The particle is projected from P (Fig. 7.3.20). It makes an angle  $45^\circ$  with the horizontal at the time of projection.



**Fig. 7.3.20**

It reaches A(x, y) at time,  $t (= v_0/g)$ . Let its velocity at A be making an angle with the horizontal.

We have,

$$x = v_0 \cos 45^\circ \cdot t$$

$$= v_0 \frac{1}{\sqrt{2}} \frac{v_0}{g} = \frac{v_0}{\sqrt{2}g}$$

$$y = v_0 \sin 45^\circ t - \frac{1}{2}gt^2$$

$$= \frac{v_0^2}{2g}(\sqrt{2} - 1)$$

$$v_x = v \cos \theta = v_0 \cos 45^\circ = \frac{v_0}{\sqrt{2}}$$

$$v_y = v \sin \theta = v_0 \sin 45^\circ - gt$$

$$= v_0 \cdot \frac{1}{\sqrt{2}} - g \cdot \frac{v_0}{g}$$

$$= \frac{v_0}{\sqrt{2}}(1 - \sqrt{2}),$$

which is  $-ve$ . It means that  $v_y$  will point vertically downward. In other words, point  $A$  is located beyond the greatest height.

The angular momentum,  $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$

$$= m (\mathbf{r} \times \mathbf{v})$$

$$= m(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \times (v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}})$$

$$\therefore \mathbf{L} = m(xv_y - yv_x) \hat{\mathbf{k}}$$

$$\mathbf{L} = m(xv)$$

$$= m \left\{ \frac{v_0^2}{\sqrt{2}g} \frac{v_0}{\sqrt{2}} (1 - \sqrt{2}) - \frac{v_0^2}{2g} (\sqrt{2} - 1) \frac{v_0}{\sqrt{2}} \right\} \hat{\mathbf{k}}$$

$$= m \frac{v_0^3}{g} \left( \frac{1-\sqrt{2}}{2} - \frac{\sqrt{2}-1}{2\sqrt{2}} \right) \hat{\mathbf{k}}$$

$$= \frac{mv_0^3}{g} \left( \frac{\sqrt{2}-2-\sqrt{2}+1}{2\sqrt{2}} \right) \hat{\mathbf{k}}$$

$$= - \frac{mv_0^3}{2\sqrt{2}g} \hat{\mathbf{k}}$$

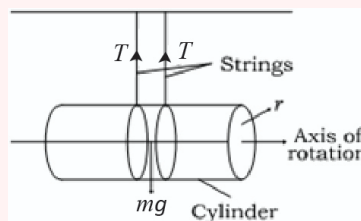
$$= \frac{mv_0^3}{2\sqrt{2}g}$$

So, the magnitude of  $\mathbf{L}$  is  $\frac{mv_0^3}{2\sqrt{2}g}$ . Its direction is along  $(-\hat{\mathbf{k}})$

is perpendicular to the  $x$ - $y$  plane pointing in the direction of the advancement of the screw-head when the screw rotates from 'x' towards 'y'. So it points towards the reader. And  $(-\hat{\mathbf{k}})$  points away from the reader into the plane of the paper. Hence, the direction of angular momentum will be perpendicular to the plane of projection pointing in the direction away from the side one observes.

**Example 7.3.6** Two strings are wound symmetrically around a cylinder of mass  $m$  and radius  $r$ . The strings are vertical when the cylinder is horizontally at rest with the strings wound around it

(Fig. 7.3.21). Then it is let go. Determine (i) the tension in the strings, (ii) the linear acceleration as the cylinder unwinds itself.



**Fig.7.3.21:** The cylinder held horizontally at rest with two strings tied symmetrically as shown.

**Solution:.**

Since the strings holding the cylinder are symmetrically disposed, the tensions in them will be equal, say  $T$ . They are two equal and parallel forces (Fig. 7.3.21). So the resultant is  $2T$  and it points vertically upwards through the centre in the direction opposite to  $mg$ .

When it starts unwinding, its downward acceleration is caused by the resultant of  $mg$  and  $2T$ .

Hence,

$$mg - 2T = ma \quad (7.3.26)$$

$$a = r\alpha \quad (7.3.27)$$

where  $a$  is the linear acceleration in downward vertical direction, and  $\alpha$  is the angular acceleration of unwinding cylinder

$$\alpha = \frac{\text{the torque}}{I},$$

$I$  is the moment of inertia of the cylinder about its axis  $= \frac{1}{2}mr^2$

Torque = The moment of the resultant,  $2T$  about the axis of rotation  $= 2Tr$

$$\therefore \alpha = \frac{2Tr}{\frac{1}{2}mr^2} = \frac{4T}{mr}$$

$$m\alpha r = 4T$$

$$ma = 4T \quad (7.3.28)$$

From Eqs(7.3.26) and (7.3.28), we get

$$mg - 2T = 4T$$

$$\text{or } 6T = mg,$$

$$T = \frac{mg}{6}$$

And from Eq(7.3.28), we get

$$a = \frac{4T}{m} = \frac{4g}{6} = \frac{2g}{3}$$

**Example 7.3.7** A string of length  $l$  is fixed at one end and carries a mass  $m$  at the other end. The string makes  $2/\pi$  revolutions per second around a vertical axis through the fixed end. Obtain the value of the angle of inclination of the string with the vertical and the linear velocity of the mass.

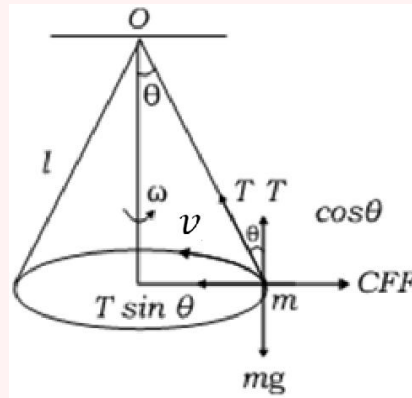
**Solution:**

The mass ' $m$ ' revolves in a circle of radius  $l \sin \theta$  (Fig. 7.3.22).

So the centrifugal force (CFF) experienced

by it =  $m\omega^2 l \sin \theta$ .

The three forces in action are  $T$ ,  $mg$  and CFF as shown in Fig.7.3.22



**Fig.7.3.22 :** The string is fixed at  $O$ . It revolves with the mass  $m$  making an angle  $\theta$  with the vertical

We have,

$$T \cos \theta = mg, \quad (7.3.29)$$

$$T \sin \theta = m\omega^2 l \sin \theta \quad (7.3.30)$$

If the linear velocity of the mass is  $v$ , then,  $\omega = \frac{v}{l \sin \theta}$  Substituting in Eq. (7.3.30) we get,

$$T \sin \theta = m \frac{v^2}{l^2 \sin^2 \theta} l \sin \theta = \frac{mv^2}{l \sin \theta}$$

Again,  $v = \omega l \sin \theta = 2\pi n l \sin \theta$ , where  $n$  = the number of revolutions per second.

$$\therefore v = 2\pi \frac{2}{\pi} l \sin \theta = 4l \sin \theta$$

$$\therefore v^2 = 16l^2 \sin^2 \theta$$

$$\begin{aligned}\therefore T \sin \theta &= \frac{m}{l \sin \theta} 16l^2 \sin^2 \theta \\ &= 16ml \sin \theta\end{aligned}$$

$$\therefore T = 16ml$$

$\therefore$  using Eq(7.3.29) we get

$$16ml \cos \theta = mg$$

$$\therefore \cos \theta = \frac{g}{16l}$$

$$\text{or, } \theta = \cos^{-1}\left(\frac{g}{16l}\right)$$

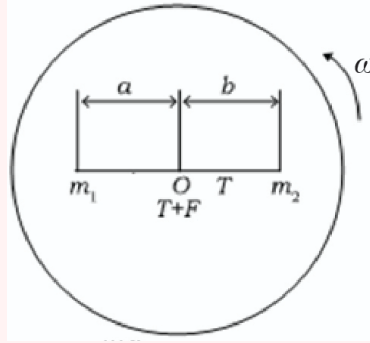
$$v = 4l \sin \theta = 4l \sqrt{1 - \cos^2 \theta}$$

$$= 4l \sqrt{1 - \frac{g^2}{256l^2}}$$

$$\therefore v = \frac{1}{4} \sqrt{256l^2 - g^2}$$

**Example 7.3.8** Two blocks of masses  $m_1$  and  $m_2$ , connected to each other by a massless inextensible string of length  $l$  are placed along a diameter of a turntable. The coefficient of friction between the table and  $m_1$  is  $\mu$ , while there is no friction between  $m_2$  and the table. The distance of  $m_1$  from the centre of the table is  $a$ . When the table is made to rotate about a vertical axis through the centre of the table with an angular velocity  $\omega$ , the masses are found to be at rest on the turn- table with respect to an observer on it. Obtain the following –

- (i) The minimum value of the angular speed of the turn-table that will make the masses slip from their positions.
- (ii) The frictional force on  $m_1$
- (iii) The ratio between the distances from the centre of  $m_1$  and  $m_2$ , so that the string remains taut and there is no frictional force on  $m_1$ .

**Solution:**

**Fig.7.3.23:** The schematic diagram of the turn-table. The masses  $m_1$  and  $m_2$  are placed at distances 'a' and 'b' from O on its two sides.

When the string remains taut, it is under tension. Let the tension be  $T$ , which means  $m_1$  will experience a force  $T$  towards its right and  $m_2$ , a force  $T$  towards its left (Fig. 7.3.23). In addition to  $T$ ,  $m_1$  also experiences a force,  $F_s$  due to friction. We shall first work out (ii). According to the observer on the turn-table,  $m_1$  and  $m_2$  are at rest. The forces on  $m_2$  are  $T$  and the centrifugal force (CFF)

$$\therefore T = m_2 \omega^2 b = m_2 \omega^2 (l - a) \quad (7.3.31)$$

The forces on  $m_1$  are  $(T + F_s)$  and the CFF,

$$\therefore T + F_s = m_1 \omega^2 a \quad (7.3.32)$$

$$\therefore F_s = m_1 \omega^2 a - T$$

Hence, from Eq(7.3.32), we get

$$F_s = m_1 \omega^2 a - m_2 \omega^2 (l - a)$$

$$\therefore F_s = \omega^2 \{m_1 a - m_2 (l - a)\}$$

We are now in a position to work out (i).

The masses will slip from their positions, when

$$\therefore F_s < \omega^2 \{m_1 a - m_2 (l - a)\}$$

They will start slipping when  $F_s$  just become less than the quantity on the RHS above. Now  $\omega$  is the only variable quantity on the RHS.

The slipping will start when  $\omega$  becomes minimum, say  $\omega = \omega_{\min}$  and  $F_s$  attains the limiting value,  $\mu m_1 g$

$$\therefore \omega_{\min}^2 \{m_1 a - m_2 (l - a)\} = \mu m_1 g$$

$$\omega_{\min}^2 = \frac{\mu m_1 g}{m_1 a - m_2 (l - a)}$$

$$\therefore \omega_{\min} = \left\{ \frac{\mu m_1 g}{m_1 a - m_2 (l - a)} \right\}^{1/2}$$

(iii) The condition for the string remaining taut but with no frictional force on  $m_1$  also is given by

$$m_1 \omega^2 a = m_2 \omega^2 b$$

$\therefore$  The required ratio,  $\frac{a}{b} = \frac{m_2}{m_1}$

**Example 7.3.9** A carpet of mass  $m$  of inextensible material is tightly rolled along its length in the form of a cylinder of radius  $r$  and is kept on a rough floor. The carpet is made to unroll without sliding on the floor by way of a negligibly small push provided to it. Calculate the horizontal velocity of the axis of the cylindrical part of the carpet when its radius becomes one-third. (Assume the mass distribution of the carpet to be uniform.)

**Solution:** When the carpet starts rolling, its CM is at a height  $r$  above the ground.

Its potential energy,  $U = mgr$

After unrolling to the extent that its radius becomes  $\frac{r}{3}$ , its potential energy now becomes

$U' = m' g \frac{r}{3}$ , where  $m'$  is the mass of the remaining part of the carpet.

$$m' = \pi \left( \frac{r}{3} \right)^2 \times l \rho,$$

where  $l$  is the width and  $\rho$  is the mass per unit area of the carpet.

$$\text{then } \frac{m'}{m} = \frac{\pi\left(\frac{r}{3}\right)^2 \times l\rho}{\pi r^2 \times l\rho} = \frac{1}{9}$$

$$\therefore m' = \frac{m}{9}$$

$$\text{Then, } U' = \frac{m}{9} \cdot g \cdot \frac{r}{3} = \frac{mgr}{27}$$

Change in potential energy

$$U - U' = mgr - \frac{mgr}{27} = \frac{26}{27}mgr \quad (7.3.33)$$

This change in potential energy is converted into kinetic energy of linear and rotational force.

$$\begin{aligned} (KE)_{rot} &= \frac{1}{2}I\omega^2 \\ &= \frac{1}{2} \cdot \left\{ \frac{1}{2}m'(r')^2 \right\} \left( \frac{v}{r'} \right)^2 \end{aligned}$$

where  $v$  = the linear velocity

$$\therefore (KE)_{rot} = \frac{1}{4} \cdot \frac{m}{9} \cdot v^2 = \frac{m}{36}v^2$$

$$\begin{aligned} (KE)_{trans} &= \frac{1}{2}m'v^2 \\ &= \frac{1}{2} \cdot \frac{m}{9} \cdot v^2 = \frac{m}{18}v^2 \end{aligned}$$

$$\therefore (KE)_{rot} + (KE)_{trans} = mv^2 \left( \frac{1}{36} + \frac{1}{18} \right) = \frac{mv^2}{12} \quad (7.3.34)$$

From Eq(7.3.33) and Eq(7.3.34), we get

$$\frac{m}{12}v^2 = \frac{26}{27}mgr$$

$$\text{Or, } v^2 = \frac{26 \times 4}{9}gr, \quad \therefore v = \frac{2}{3}\sqrt{26gr}$$

### 7.3.11 Summary

- The rotational motion of a body can be considered as the rotation of all the particles constituting the body about the same axis.
- The torque of a force plays a cardinal role in the study of dynamics of rotational motion and it is given by the vector product of the radius vector of the point of application of the force with the force itself.
- Angular momentum is defined as the moment of linear momentum and it is obtained as the vector product of the radius vector and linear momentum.
- In determination of the torque and the angular momentum, locating the radius vector, also called the lever-arm, is a crucial factor.
- Inertia in linear motion is represented by a quantity called moment of inertia. For a point mass the moment of inertia is the product of the mass and the square of its distance from the axis of rotation. For a system of point masses, it is the summation of the above product over all such points.
- The radius of gyration of a body about one axis is the square root of the ratio of the moment of inertia to the mass of the body.
- The analogies between linear and rotational motions are given in the Table below (symbols have their usual meanings) .

| Linear Motion                                 | Rotational Motion                                       |
|-----------------------------------------------|---------------------------------------------------------|
| Linear Displacement ( $r$ )                   | Angular Displacement( $\theta$ )                        |
| Linear Velocity( $v$ )= $\frac{dr}{dt}$       | Angular Velocity ( $\omega$ ) = $\frac{d\theta}{dt}$    |
| Linear Acceleration ( $a$ ) = $\frac{dv}{dt}$ | Angular Acceleration( $\alpha$ ) = $\frac{d\omega}{dt}$ |
| Force ( $F$ )                                 | Torque ( $\tau = \mathbf{r} \times \mathbf{F}$ )        |
| Inertia ( $m$ )                               | Moment of Inertia ( $I = \sum mr^2$ )                   |

|                                             |                                                        |
|---------------------------------------------|--------------------------------------------------------|
| Linear Momentum ( $p = mv$ )                | Angular Momentum ( $\mathbf{L} = r \times p$ )         |
| Linear Displacement ( $r$ )                 | Angular Displacement ( $\theta$ )                      |
| Linear Velocity ( $v = \frac{dr}{dt}$ )     | Angular Velocity ( $\omega = \frac{d\theta}{dt}$ )     |
| Linear Acceleration ( $a = \frac{dv}{dt}$ ) | Angular Acceleration ( $\alpha = \frac{d\omega}{dt}$ ) |

- If the resultant torque acting on a body or a system of particles is zero, then its angular momentum is conserved.
- For a body rolling without slipping,  $V_{CM} = r\omega$ , with symbols having usual meaning.
- There can be three kinds of equilibrium of a body-
  - (a) Stable equilibrium (when potential energy is minimum)
  - (b) Unstable equilibrium (when potential energy is maximum)
  - (c) Neutral equilibrium (when potential energy is constant)

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