

Module 1 Simple Harmonic Motion

This module is designed to help you to:

- identify motions accompanied with vibrations and find out those among them which exhibit uniform periodicity;
- identify oscillatory motion as a particular case of periodic motion;
- describe the characteristics of simple harmonic motion;
- correlate simple harmonic motion with uniform circular motion;
- obtain an expression for instantaneous displacement in simple harmonic motion;
- explain the meaning of terms - phase and phase difference.

Content

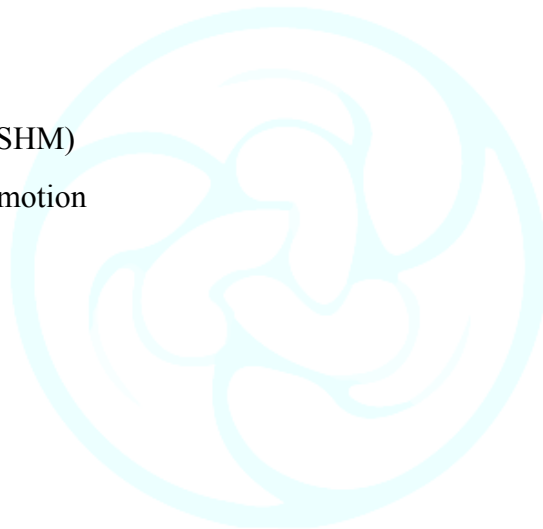
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14.1.1 Introduction

Motion is an integral part of our existence. We experience various kinds of motion around us, and we take interest in analyzing and understanding motions. One particular type of motion is a periodic motion, which repeats itself. The time after which motion repeats itself may be a constant or may vary. Some common examples of periodic motion are:

- (i) motion of hands of a clock
- (ii) motion of earth around its own axis
- (iii) motion of a mass attached to a spring
- (iv) motion of a simple pendulum
- (v) variation of alternating voltage.

An important type of periodic motion is oscillatory motion. In oscillatory motion, the motion repeats itself about (to and fro) a mean or equilibrium position.

Every periodic motion is not necessarily oscillatory motion. Examples (i) and (ii) are periodic but not oscillatory. Examples (iii), (iv) and (v) are periodic as well as oscillatory. Numbers (iii) and (iv) are examples from mechanical systems, whereas number (v) is about an electrical system. In this module we shall restrict our discussions to mechanical cases only.

14.1.2 Simple Harmonic Motion (SHM)

A very important and useful type of oscillatory motion is Simple Harmonic Motion (SHM). In the case of SHM, the total mechanical energy remains conserved. The energy of a particle or a body executing SHM alternates between potential and kinetic form, and the motion is supposed to continue forever in absence of dissipative forces. SHM can be defined in several ways as under -

- (i) The displacement vs. time curve of the particle is sinusoidal, i.e., either a sine curve or a cosine curve.
- (ii) The potential energy of the particle is proportional to the square of its displacement.

(iii) The force experienced by the particle is directly proportional to its displacement and it acts towards the equilibrium or mean position.

We shall start our discussion with Definition No. (iii).

We see that the force is directed towards the mean position, which means that it is a restoring force, i.e. the force tries to bring the object back towards its equilibrium (or mean) position; and the magnitude of force is directly proportional to the instantaneous displacement. i.e.

$$|F| \propto x$$

Expressed mathematically

$$F = -kx \quad (14.1.1)$$

k is a constant of proportionality. It is known as the force constant or the spring factor. Negative sign indicates restoring nature of the force.

Consider a particle P executing SHM along the X -axis (Refer to Fig 14.1.1). O is the mean position. L and M are the extreme positions around O . $OL = OM = A$, which is called the amplitude of S.H.M

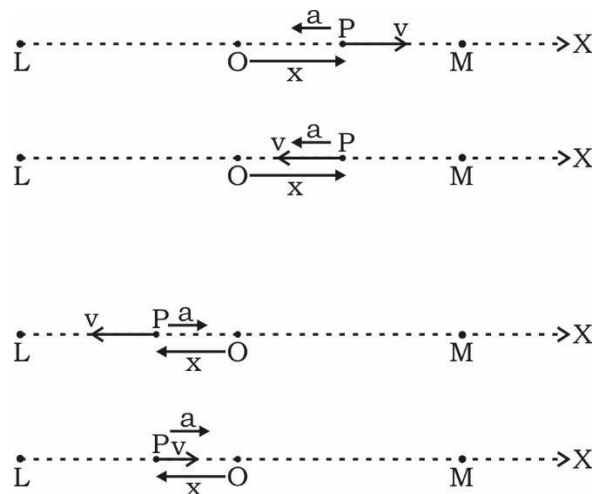


Fig. 14.1.1

Fig. 14.1.1: A particle P executing SHM at different points about mean position O .

Note: 1. The arrow of the acceleration (a) is always directed towards the mean position O .

2. The arrows of displacement (x) and acceleration (a) are always directed opposite to one another. This is expressed by the negative sign in Eq. (14.1.1).

Consider a particle of mass m executing SHM around mean position **O** along the X- axis. Let x be the displacement of m at any instant $t = t$.

The equation of motion is $m \frac{d^2x}{dt^2} = - kx$.

The above is a second order differential equation in x and t

k is the force constant

So, its dimension = $\frac{[Force]}{[Length]} = \frac{MLT^{-2}}{L} = MT^{-2}$.

$\therefore$ Dimension of $\frac{k}{m} = T^{-2} = \left(\frac{1}{T}\right)^2$

So, we replace $\frac{k}{m}$ by ω^2 , where ω is a constant whose dimension is $\frac{1}{T}$, i.e. reciprocal

of time. So, we get the differential equation as under.

$$\frac{d^2x}{dt^2} = - \omega^2 x \quad (14.1.2)$$

ω turns out to be the angular frequency of SHM, i.e. $\omega = 2\pi\nu = \frac{2\pi}{T}$, where ν is the natural

frequency and T is the time period.

It can be shown that the time period (T) of the motion is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

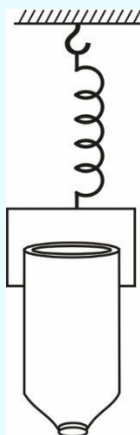
Activity 14.1.1

Fig. 14.1.2: *A helical spring fixed in vertical position*

Let us take a helical spring and fix it up in a vertical position as shown in Fig.14.1.2. In place of the mass attached at the free end of the spring, let us suspend a PET bottle with its mouth in the downward direction. If we now fill up the bottle with water and pull it a little bit and then release it, the bottle will oscillate. Note down the time period of oscillation. Now, take out about half the water from the bottle and repeat the experiment. The spring will oscillate with a lower time period. Since the spring constant (k) is the same in both the cases the higher mass gives a longer time period of oscillation and a smaller mass gives a shorter time period of oscillation.

Now, let us replace the cap with another cap with a small hole. Starting with a completely filled bottle; observe how the time period of oscillation varies as the mass of the water in the bottle decreases continuously till the bottle is completely empty.



Let Us Reflect

Q.14.1.1 x , v and f represents respectively the displacement, velocity and acceleration of a particle. Given below are four equations {(i), (ii),(iii),(iv)} connecting these variables.

(i) $f + 12x = 0$

(ii) $f + 2v + 8x = 0$

(iii) $v^2 = x^2 - 6x - 8$

(iv) $v^2 = 16 + 6x - x^2$

Pick up the ones from the above which represent SHM. In such cases(s) determine the value(s) of the time periods.



Answers

A.14.1.1 Written in the form of differential equation (i) and (ii) take the following shapes:

$$\frac{d^2x}{dt^2} + 12x = 0 \quad (14.1.3)$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 8x = 0 \quad (14.1.4)$$

Eq((14.1.3) is similar to Eq(14.1.2) with $\omega^2 = 12$.

So, it represents SHM.

$$\omega = \sqrt{12} = 2\sqrt{3} \text{ sec}^{-1}$$

$$\therefore \text{Time period} = \frac{2\pi}{\omega} = \frac{\pi}{\sqrt{3}} \text{ sec.}$$

Because of the term $2\frac{dx}{dt}$, Eq(14.1.4) is not of the shape of Eq(14.1.2). So Eq(14.1.4) does not represent SHM.

For (iii), we have

$$\frac{d}{dx}(v^2) = 2x - 6 = 2(x - 3)$$

$$2v\frac{dv}{dx} = 2(x - 3)$$

$$\therefore \frac{dx}{dt} \frac{dv}{dx} = \frac{dv}{dt} = x - 3$$

$$\text{Or, } f = x - 3$$

$$\text{i.e } \frac{d^2x}{dt^2} = x - 3, \text{ which is not the differential equation of SHM.}$$

For (iv), we have

$$2v\frac{dv}{dx} = -2x + 6 = -2(x - 3)$$

$$\therefore f = -x + 3$$

$$\text{Or } f + (x - 3) = 0$$

Putting $x - 3 = x'$, we get

$$\frac{d^2x'}{dt^2} + x' = 0$$

It is the differential equation of SHM,

with $\omega^2 = 1$, or $\omega = 1 \text{ sec}^{-1}$

$\therefore$ Time period = $T = \frac{2\pi}{\omega} = 2\pi \text{ sec}$

So (i) and (iv) represent SHM and the respective values of Time period are $\frac{\pi}{\sqrt{3}} \text{ sec}$ and $2\pi \text{ sec}$

14.1.3 SHM and uniform circular motion

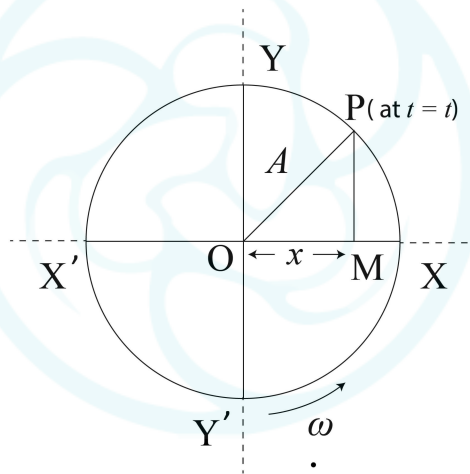


Fig. 14.1.3: Diagram to relate SHM and uniform circular motion

An SHM can be expressed in terms of a projection at an instant of time of the radius vector of a particle executing uniform circular motion along any diameter of the circle. Consider a particle P in uniform circular motion with angular velocity ω along a circle of radius $OP = A$. The particle starts from point X at $t = 0$. P represents the position of the particle at $t = t$. The projection of OP along diameter $X'OX$ is OM . Let us now analyze the motion of M . As P moves from X to Y ; the point M moves from X to O . As P moves from Y to X' ; M moves from O to X' . Similarly, during the motion of the particle from X' to Y' and finally to X ; M moves from X' to O and finally back to X . Let us now observe the motion of M about O .

$$OM = OP \sin \angle POM$$

If the angular velocity is ω , the angle POM has been described in time t , then $\angle POM = \omega t$

if $OM = x$, we have

$$x = A \sin \omega t$$

$$\therefore \frac{dx}{dt} = A\omega \cos \omega t,$$

$$\text{and } \frac{d^2x}{dt^2} = -A\omega^2 \sin \omega t = -\omega^2 x,$$

which is exactly the same as Eq(14.1.2). So the point M executes SHM. with an angular frequency ω , which is the same as the angular velocity of the particle executing uniform circular motion.

Similarly, the feet of the perpendiculars of the position of such particles on any other diameter execute SHM about O.

T = Time period of SHM

= Time taken by P to complete one full circle with constant angular velocity ω .

$$= \frac{2\pi}{\omega}$$

The maximum displacement of M, ($OX = OX'$) is the amplitude of motion and the amplitude equals the radius of the circle of reference (i.e. the circle along which P moves).

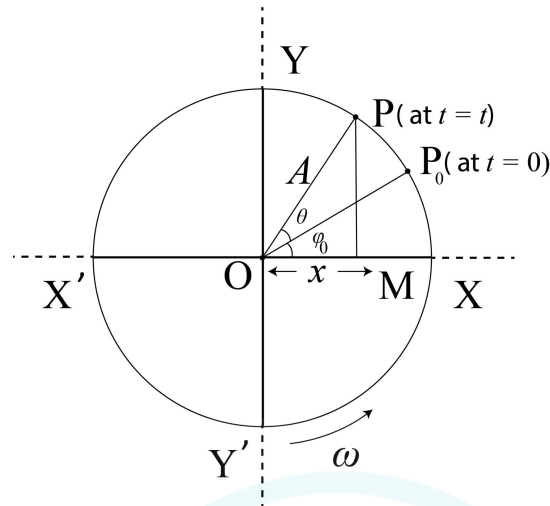


Fig. 14.1.4

In Fig. (14.1.4) P_0 is the position of a particle moving along the circle of reference of radius $OP = A$ at $t = 0$. $\angle XOP_0 = \phi_0$, P is the instantaneous position of the particle on its circular path and M is the instantaneous projection of P along diameter XOX' . M is executing SHM. Obviously $OM = x$ is the instantaneous displacement of the particle and

$$\theta = \omega t; \angle MOP = \theta + \phi_0 = \omega t + \phi_0$$

From right angled $\triangle OMP$,

$$OM = x = OP \cos (\theta + \phi)$$

$$x = A \cos (\omega t + \phi_0) \quad (14.1.5)$$

where $OP = A = \text{Amplitude of SHM}$

Eq(14.1.5) is the equation of instantaneous displacement. Now it can be appreciated that the displacement vs. time equation of an SHM is sinusoidal, which corresponds to Definition (i) of SHM. Here, we have got a cosine curve. It can be a sine curve as well. It can also be obtained by way of solving the differential equation (14.1.2).

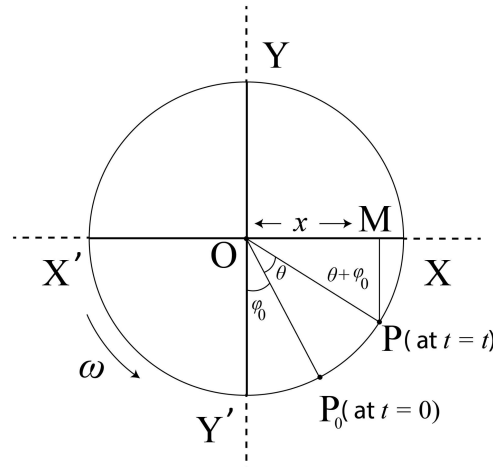


Fig. 14.1.5 a particle P moving along a circle of reference as described earlier. The angular position of P is measured with reference YOY' -axis instead of XOX' -axis as described earlier.

$OM = x =$ instantaneous displacement of M

$$= OP \sin \angle OPM$$

$$= A \sin (\theta + \phi_o) = A \sin (\omega t + \phi_o)$$

Equations like

$$x = A \sin (\omega \sqrt{t} + \phi_o),$$

$$x = A \log [\sin (\omega t)]$$

or $x = A e^{-\alpha t}$; where α is a constant

DO NOT represent SHM. On differentiating any of the above expressions the differential equation of SHM is NOT satisfied. Hence these do not represent SHM.



Let Us Reflect

Q.14.1.2 A particle is executing uniform circular motion such that the angle described by it per unit time is 60° with respect to a point on the circumference of the circle. What will be the time period of simple harmonic oscillations executed by the foot of the perpendicular drawn from the position of the particle on any diameter of the circle?



Answers

A.14.1.2 The angular frequency of the oscillation is equal to the angular velocity of the particle (measured in radians/sec) about the center of the circle.

The angle described per unit time about the centre

$$= 2 \times 60^\circ = \frac{2\pi}{3} \text{ radians}$$

$$\therefore \frac{2\pi}{3} \text{ radians s}^{-1}$$

$$\therefore T = \frac{2\pi}{\frac{2\pi}{3}} = 3 \text{ sec}$$

Example 14.1.1

Draw the Circle of Reference (CoR) for each of the following SHM. What is (i) the initial ($t=0$), position, (ii) the radius of the Circle of Reference and (iii) the angular speed of the particle, moving in anticlockwise direction, along the Circle of Reference. (x is in cm and t is in second).

$$(a) \ x = -2 \sin\left(3t + \frac{\pi}{3}\right)$$

$$(b) \ x = \cos\left(\frac{\pi}{6} - t\right)$$

$$(c) \ x = 3 \sin\left(2\pi t + \frac{\pi}{4}\right)$$

$$(d) \ x = 2 \cos \pi t$$

Solution

$$(a) \ x = -2 \sin\left(3t + \frac{\pi}{3}\right)$$

$$= 2 \cos\left(3t + \frac{\pi}{3} + \frac{\pi}{2}\right)$$

$$\because \cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$$

$$\text{or } x = 2 \cos\left(3t + \frac{5\pi}{6}\right)$$

Let us compare this equation with one of the standard equation of a SHM

$$\text{i.e. } x = A \cos(\omega t + \phi) \quad (14.1.4)$$

We find that here, $A = 2 \text{ cm}$, $\omega = 3 \text{ radian s}^{-1}$. and $\phi = \frac{5\pi}{6} = 150^\circ$

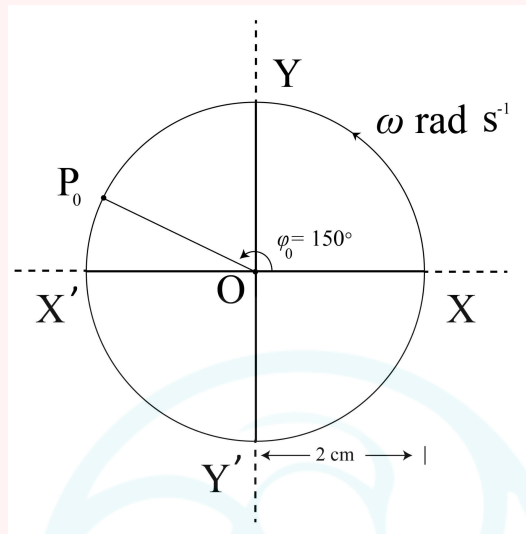


Fig. 14.1.6

CoR of the given SHM is shown in **Fig. 14.1.6**

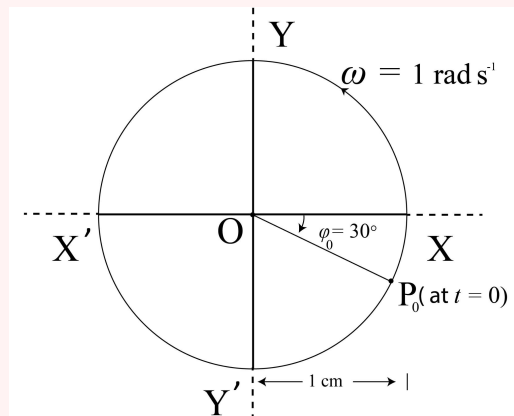
$$\begin{aligned} \text{(b) } x &= \cos\left(\frac{\pi}{6} - t\right) [\because \cos\theta = \cos(-\theta)] \\ &= \cos\left(t - \frac{\pi}{6}\right) \end{aligned}$$

Comparing the given equation with Eq(14.1.4); we have

$$A = 1 \text{ cm}, \omega = 1 \text{ radian s}^{-1}$$

$$\phi = -\left(\frac{\pi}{6}\right) = -30^\circ$$

The CoR of this SHM is shown in Fig. 14.1.7.

**Fig. 14.1.7**

Similarly, Reference Circles for cases (c) and (d) can be drawn.

14.1.4 Phase in SHM

The phase of SHM at any time $t = t$; is a term or quantity that gives complete information regarding the state of a particle (that is, displacement, velocity, acceleration) at that moment of time.

In the next module we shall derive the velocity and acceleration of a particle executing SHM, and we shall see from the graphical representations of the variation of these parameters with time that the lead or lag of a parameter with reference to another is governed by the argument part of the sine or cosine function concerned. Amplitude does not have any role there. So, the argument part of the sinusoidal function is the measure of phase.

For SHM described by the equation

$$x = A \sin (\omega t + \phi_o)$$

the instantaneous phase (ϕ) is

$$\phi = \omega t + \phi_o$$

ϕ at $(t = 0) = \phi_o$ which is the initial phase or epoch of SHM.

$\Delta\phi$ = Phase difference of SHM at time $t = t_1$ and at time $t = t_2$

$$= [\omega(t_2) + \phi_o] - [\omega(t_1) + \phi_o]$$

$$= \omega(\Delta t) = \frac{2\pi}{T}(\Delta t) \quad ; \text{ where } \Delta t = t_2 - t_1$$

For the two positions in phase, the displacement, velocity and acceleration are same in magnitude and direction if

$$\Delta t = nT$$

Or $\Delta\phi = 2nT\pi$, where $n = 0, 1, 2, 3, \dots$

However, if displacement, velocity and acceleration are same in magnitude but opposite in direction; the two positions are out of phase with one another. This occurs at

$$\Delta t = (2n-1)\frac{T}{2}$$

Or $\Delta\phi = (2n-1)\pi$, where $n = 1, 2, 3, \dots$

For two SHMs described by

$$x_1 = A_1 \sin(\omega_1 t + \phi_{10}) \quad \text{and}$$

$$x_2 = A_2 \sin(\omega_2 t + \phi_{20})$$

the instantaneous phase difference ($\Delta\phi$) between the two SHM's is

$$\Delta\phi = (\omega_1 - \omega_2)t + (\phi_{10} - \phi_{20})$$

Case I $\omega_1 = \omega_2$

$$\Delta\phi = \phi_{10} - \phi_{20} = \Delta\phi_o = \text{constant}$$

The phase difference is the same as the initial phase difference at all times.

Case II $\omega_1 \neq \omega_2$

Let $\phi_{10} = \phi_{20}$ (i.e. initial phase difference = 0). Then

$$\Delta\phi = (\omega_1 - \omega_2)t$$

Note $\Delta\phi$ varies with time. There are some moments of time when the two particles are in phase; and some other moments of time when the two are out of phase with one another.

Example 14.1.2 A simple harmonic motion is described by the equation

$$x = A \sin(\omega t + \frac{\pi}{4})$$

What is the phase difference between two positions of the particle at, given by

- (i) $t = \frac{T}{3}$ and $t = \frac{T}{2}$
 (ii) $x = \frac{A}{\sqrt{2}}$ and $x = \frac{A}{2}$

Solution:

- (i) The phase of the particle at any time $t = t$

$$\phi = \omega t + \frac{\pi}{4}$$

$$\begin{aligned} \text{At } t = \frac{T}{3}; \phi_1 &= \omega \frac{T}{3} + \frac{\pi}{4} \\ &= \frac{2\pi}{T} \times \frac{T}{3} + \frac{\pi}{4} = \frac{11}{12}\pi \end{aligned}$$

$$\text{At } t = \frac{T}{2}; \phi_2 = \omega \frac{T}{2} + \frac{\pi}{4} = \frac{5}{4}\pi$$

$$\therefore \text{Phase difference} = \phi_2 - \phi_1 = \left(\frac{5\pi}{4} - \frac{11\pi}{12} \right) = \frac{\pi}{3}$$

(ii) At $x = \frac{A}{\sqrt{2}}$ & $\frac{A}{2}$, we have the phases as $\sin^{-1}(\frac{1}{\sqrt{2}})$ & $\sin^{-1}(\frac{1}{2})$, i.e., $\frac{\pi}{4}$ & $\frac{\pi}{6}$

$$\text{So, the reqd phase difference} = \left[\frac{\pi}{6} - \frac{\pi}{4} \right] = - \left(\frac{\pi}{12} \right)$$

Thus, the latter is ahead of the former by $\frac{\pi}{12}$.

Example 14.1.3 An a. c. voltage is described by the equation

$$V(t) = V_0 \sin(\omega t + \phi)$$

where V_0 is the peak value, and ω the angular frequency of a.c.

(a) For an a.c. of frequency 50 Hz, peak value 200 V, write the equation of a.c.

if at $t = 0$; $V(0)$ is

- (i) 0 V (ii) 100 V (iii) 200 V

(b) What is the instantaneous voltage in each case at $t = \frac{3}{4}$ s?

Solution:

(a) The general equation of harmonic variation of a.c. voltage is

$$\begin{aligned} V(t) &= V_0 \sin(2\pi\nu t + \phi_0) \\ &= 200 \sin(100\pi t + \phi_0) \end{aligned} \tag{14.1.5}$$

(i) At $t = 0$; $V = 0$, therefore

$$\sin \phi_0 = 0$$

$$\therefore \phi_0 = 0$$

The Eq(14.1.5) reduces to

$$V = 200 \sin(100\pi t)$$

(ii) At $t = 0$; $V = 100$ V, therefore

$$200 \sin \phi_0 = 100 \text{ or } \sin \phi_0 = \frac{1}{2}$$

$$\therefore \phi_0 = \pi/6$$

Substituting value of ϕ_0 in Eq (14.1.5) ; we have

$$V = 200 \sin(100 \pi t + \frac{\pi}{6})$$

(iii) At $t = 0$; $V = 200$ V from Eq(14.1.5) we get

$$200 = 200 \sin \phi_0$$

$$\therefore \sin \phi_0 = 1 \text{ and } \phi_0 = \frac{\pi}{2}$$

$$\therefore V = 200 \sin (100 \pi t + \frac{\pi}{2})$$

$$= 200 \cos(100 \pi t)$$

(b) Given $t = \frac{3}{4}$ s; therefore

$$\text{Case (i) } V = 200 \sin (100 \pi t)$$

$$= 200 \sin 100 \pi (\frac{3}{4}) = 200 \sin 75\pi = 0, \text{ because } \sin(n\pi) = 0, \text{ for}$$

$$n = 1, 2, 3, \dots$$

$$\text{case (ii) } V = 200 \sin (100\pi t + \frac{\pi}{6})$$

$$V = 200 \sin (100\pi \times \frac{3}{4} + \frac{\pi}{6})$$

$$= -200 \sin (75\pi + \frac{\pi}{6}) = 200 \sin(\pi + \frac{\pi}{6})$$

$$= -200 \sin(\frac{\pi}{6}) = -100 \text{ V}$$

$$\begin{aligned}\text{case (iii)} \quad &= 200 \cos\left[100\pi\left(\frac{3}{4}\right)\right] = 200 \cos(75\pi) \\ &= 200 \cos \pi = -200 \text{ V}\end{aligned}$$

Let Us Reflect

Q.14.1.3 The motion of a simple pendulum is described by the equation, $\theta = \theta_0 \sin \omega t$.

where the symbols have their usual meanings. What is the phase difference between two positions of the pendulum defined by $\theta = \theta_0$ and $\theta = \frac{\theta_0}{2}$



Answers

A.14.1.3

For $\theta = \theta_0$, $\sin \omega t = 1$,

$$\text{Or, } \omega t = \frac{\pi}{2}$$

And for $\theta = \frac{\theta}{2}$, $\sin \omega t = \frac{1}{2}$

$$\text{Or, } \omega t = \frac{\pi}{6}$$

So the required phase difference

$$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

**14.1.5 Summary**

- A periodic motion repeats itself. The time after which motion repeats may be a constant or may vary.
- To and fro motion about a mean (or equilibrium) position is an oscillatory motion.
- Simple Harmonic Motion (SHM) is an important type of periodic motion characterized by the displacement of the particle concerned being a sine or a cosine function of time.
- In SHM, the force is (i) directly proportional to the instantaneous displacement (x); and (ii) always directed towards the mean position.
- For a particle of mass m , moving along X-axis; the differential equation of SHM is

$$m \frac{d^2 x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2 x}{dt^2} = -\omega^2 x \quad ; \quad \omega^2 = \frac{k}{m}$$

where ω is known as the angular frequency of SHM.

- The foot of the perpendicular on any diameter of a circle (known as Circle of Reference) along which a particle is undergoing uniform circular motion executes SHM. The radius of the circle of reference equals amplitude (maximum displacement) of motion. The time-period, T , of SHM is given by

$$T = \frac{2\pi}{\omega}$$

where, ω = uniform angular speed of the particle moving along the circle of reference.

- The instantaneous displacement(x) vs time(t) equation of a particle executing SHM is

$$x = A \sin(\omega t + \phi_o) \text{ or } A \cos(\omega t + \phi_o)$$

- The instantaneous phase, ϕ ; of SHM is $\phi(t) = \omega t + \phi_o$ or $(\omega t + \phi_o)$
- Phase gives complete information regarding the state of the particle, concerning its displacement, velocity and acceleration at that point of time.
- For two SHM's of the same frequency ; the instantaneous phase difference ($\Delta\phi$) equals the initial phase difference between the two SHMs.
- For two SHM's of different frequencies the phase difference ($\Delta\phi$) varies with time.
- The two positions for which the particles are

(a) in phase, we have $\Delta\phi = 2n\pi$; $n = 0, 1, 2, 3, \dots$

(b) out of phase, we have $\Delta\phi = (2n - 1)\pi$; $n = 1, 2, 3, \dots$

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