

Module 2 Characteristics of Simple Harmonic Motion

This module is designed to help you to:

- explain the variation of displacement, velocity and acceleration with time and the graphical representations of such variations;
- obtain the expressions for instantaneous (i) kinetic energy (ii) potential energy and their variation with time;
- interpret that total energy of SHM is independent of (i) time and (ii) instantaneous displacement of the particle;
- correlate (i) maximum kinetic energy (ii) maximum potential energy and total energy of SHM. in the light of principle of conservation of energy;
- evaluate average (i) kinetic energy (ii) potential energy and obtain their relation with total energy of SHM.

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14.2.1 Introduction

We got introduced to the concept of simple harmonic motion through the previous module. In this module we shall study the variation with time of displacement, velocity, acceleration and energy of particles. We shall also study the graphs of the above said variation and thereby we shall also be able to appreciate that for a particle executing an SHM, the total mechanical energy ($KE+PE$) is a conserved quantity. We will also derive the values of average KE and PE during a cycle and obtain their physical significance.

14.2.2 Phase relationship between displacement, velocity and acceleration

For a particle executing SHM of amplitude (A); angular frequency(ω) having an initial phase ϕ_o , the instantaneous displacement (x) is

$$x = A \sin(\omega t + \phi_o) \quad (14.2.1)$$

The instantaneous velocity (V) of the particle is

$$\begin{aligned} V &= \frac{dx}{dt} = A \cos(\omega t + \phi_o) \\ &= V_o \cos(\omega t + \phi_o), \end{aligned} \quad (14.2.2)$$

where $V_o = A\omega$

V_o is the velocity amplitude and is obviously the maximum value of V . We can rewrite Eq(14.2.2) as

$$V = V_o \sin[(\omega t + \phi_o) + \frac{\pi}{2}] \quad (14.2.3)$$

Comparing Eqs(14.2.1) and (14.2.3); the velocity leads displacement by a constant phase of $\frac{\pi}{2}$. Eqs(14.2.2) or (14.2.3) describes V as a function of time (t). The relation between instantaneous displacement (x) and instantaneous velocity (V) is obtained using Eqs(14.2.1) and (14.2.2). Obviously

$$\begin{aligned} V &= A\omega[1 - \sin^2(\omega t + \phi_o)]^{\frac{1}{2}} \\ &= \omega[A^2 - A^2 \sin^2(\omega t + \phi_o)]^{\frac{1}{2}} \end{aligned}$$

$$= \omega[A^2 - x^2]^{\frac{1}{2}} \quad (14.2.4)$$

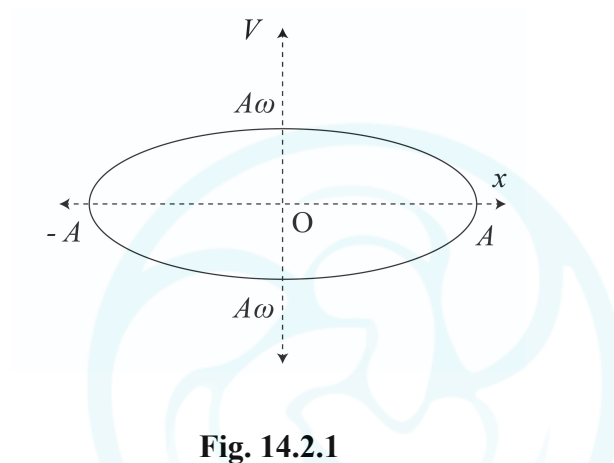
Eq(14.2.4) can be rearranged as

$$\left(\frac{v}{\omega}\right)^2 + x^2 = A^2$$

or

$$\frac{x^2}{A^2} + \frac{v^2}{(A\omega)^2} = 1 \quad (14.2.5)$$

Eq(14.2.5) is the equation of an ellipse. Graph of V versus x is an ellipse as shown in Fig. 14.2.1



The instantaneous acceleration a of the particle is

$$a = \frac{dv}{dt} = -A\omega^2 \sin(\omega t + \phi_0) \quad (14.2.6)$$

$$= a_0 \sin[(\omega t + \phi_0) - \pi] \quad (14.2.7)$$

where $a_0 = A\omega^2$. a_0 is the acceleration amplitude. Eq(14.2.6) is rewritten as

$$a = -\omega^2 x \quad (14.2.8)$$

From Eqs(14.2.1) and (14.2.7) we note that acceleration lags behind displacement by π (or acceleration is out of phase with displacement). From Eqs(14.2.3) and (14.2.7) we note that velocity leads acceleration by $\frac{3\pi}{2}$.

The variation of displacement (x), velocity (V) acceleration (a) with time (t) taking $\phi_0 = 0$ are shown in Fig 14.2.2 (a), (b) and (c). The phase relation between the different factors can be obtained from the graphs also.

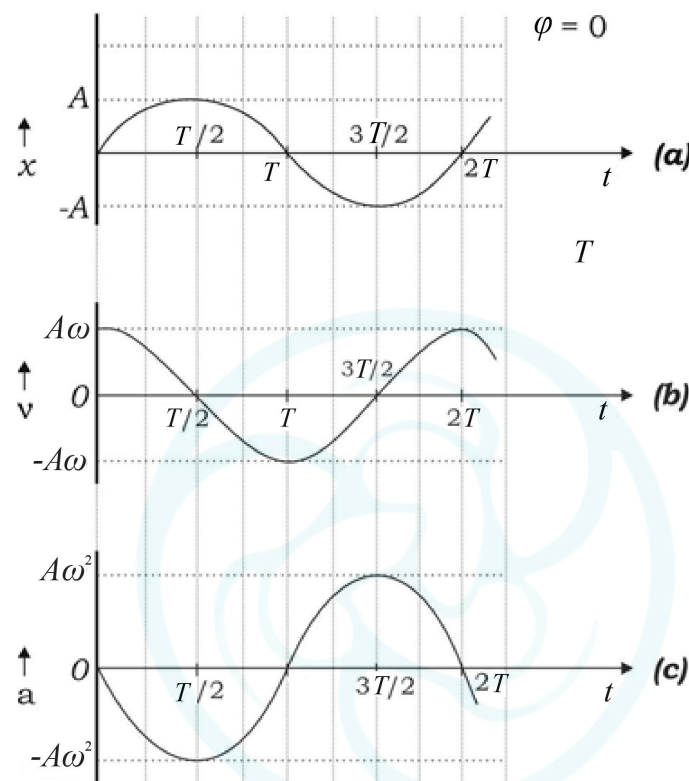


Fig. 14.2.2

The instantaneous acceleration in SHM is directly proportional to magnitude of instantaneous displacement. The graph of a versus x is a straight line with a negative slope as shown in Fig. 14.2.3.

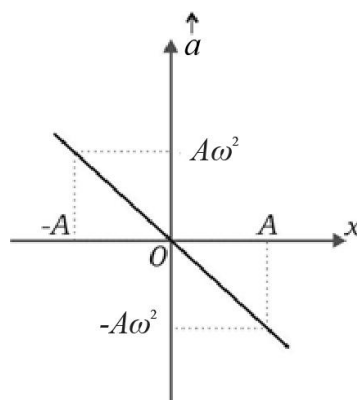


Fig. 14.2.3



Let Us Reflect

Q.14.2.1 A particle is executing SHM along a line. its velocity at two positions at distance 1 cm and 2 cm from the mean position are 4 cm/s and 3 cm/s respectively. What is the time period of oscillation ?

Q.14.2.2 A simple pendulum of length 10 m is executing simple harmonic oscillation with an angular amplitude of 18 degrees. Taking $g = 10 \text{ m/s}^2$ and $\pi = 3$, obtain the expression for angular acceleration in radians per second squared of the pendulum bob as a function of time t .

Q.14.2.3 The phase relations between displacement, velocity and displacement . acceleration are primarily based on Trigonometrical equation concerning Associate Angle, such as $\sin (90^\circ + \theta) = \cos \theta$, $\sin (90^\circ - \theta) = \cos \theta$, $\sin (180 - \theta) = \sin \theta$ and so on; tangent and cotangent function also follow similar relation such as $\tan (90^\circ - \theta) = \cot \theta$, $\tan (90^\circ + \theta) = -\cot \theta$, $\tan (180^\circ + \theta) = \tan \theta$ and so on. Further, the most fundamental relation behind periodicity, i.e., $f(2\pi + \theta) = f(\theta)$, is satisfied by all trigonometric functions, then what is the reason behind 'tan' and 'cot' functions not figuring among the ones used for understanding the variations of displacement , velocity and acceleration with time ?



Answers

A.14.2.1

With symbols having their usual meaning. we have

$$v^2 = \omega^2 (A^2 - x^2)$$

$$\text{let } v = v_1 \text{ at } x = x_1$$

$$\text{and } v = v_2 \text{ at } x = x_2$$

From Eq(14.2.1), we have

$$\frac{v^2}{\omega^2} + x^2 = A^2$$

$$\therefore \frac{v_1^2}{\omega^2} + x_1^2 = \frac{v_2^2}{\omega^2} + x_2^2$$

$$\text{or } \frac{1}{\omega^2} (v_1^2 - v_2^2) = x_2^2 - x_1^2$$

$$\text{or } \omega^2 = \frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}$$

$$\therefore \omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}$$

$$\therefore T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$$

From given data

$$v_1 = 4, \text{ when } x_1 = 1$$

$$\text{and } v_1 = 3, \text{ when } x_1 = 2$$

$$\begin{aligned} \therefore T &= 2\pi\sqrt{\frac{4-1}{16-9}} = 2\pi\sqrt{\frac{3}{7}} = (2\sqrt{\frac{3}{7}})\pi \text{ sec} \\ &= 4.1 \text{ sec} \end{aligned}$$

A.14.2.2

With symbols having their usual meaning, we have the angular displacement vs. time relation as

$$\theta = \theta_0 \sin \omega t$$

$$\therefore \frac{d\theta}{dt} = \theta_0 \omega \cos \omega t$$

$$\text{and } \frac{d^2\theta}{dt^2} = -\omega^2 \theta_0 \sin \omega t; \theta_0 = 18 \text{ degree} = \frac{\pi}{10} \text{ radian}$$

$$\therefore \alpha = -\frac{g}{l} \left(\frac{\pi}{10} \right) \sin \sqrt{\frac{g}{l}} t$$

$$\text{or } \alpha = -\frac{3}{10} \sin \sqrt{\frac{g}{l}} t$$

$$\text{or } \alpha = - (0.3) \sin \left(\sqrt{\frac{10}{10}} t \right)$$

$$\therefore \alpha = - (0.3) \sin t$$

A.14.2.3 ‘tan’ and ‘cot’ functions follow periodicity like the sine and cosine function, but they do not have finite turning points like the sinusoidal (sine and cosine) function.

14.2.3 Energy in SHM

For a particle of mass m executing SHM the displacement (x) and velocity (V) vary with time, therefore the kinetic energy (K) and potential energy (U) also vary with time. However the total energy (E) = $K+U$ is a constant. At this stage, recall the statement you have read in Module 1 - ‘The energy alternates between potential and kinetic forms’. For the SHM described by the equation, $x = A\sin(\omega t + \phi_o)$,

K = The instantaneous kinetic energy

$$\begin{aligned} &= \frac{1}{2}mV^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \\ &= \frac{1}{2}mA^2\omega^2 \cos^2(\omega t + \phi_o) \\ &= K_m \cos^2(\omega t + \phi_o) \end{aligned} \quad (14.2.9)$$

where, $K_m = \frac{1}{2}mA^2\omega^2$ is the maximum value of kinetic energy . It is also referred to as amplitude of oscillations of kinetic energy.

The instantaneous potential energy (U); by definition, equals work done in displacing particles from mean position to instantaneous position of displacement x .

$$\therefore U - U_o = \int_0^x kx dx = \left(\frac{kx^2}{2}\right)$$

or

$$U = U_o + \frac{kx^2}{2}$$

U_o is the potential energy of SHM in equilibrium (or mean position). Commonly, U_o is taken as zero.

Choosing $U_o=0$,

$U = \frac{1}{2}kx^2$. Thus, we can corroborate with the Definition No. (ii) of SHM, given at Module 1.

$$\text{so, } U = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \quad \left[\omega^2 = \frac{K}{m}\right]$$

$$= U_m \sin^2(\omega t + \phi_o) \quad (14.2.10)$$

where $U_m = \frac{1}{2} m A^2 \omega^2$ is the maximum potential energy of SHM. It is also known as amplitude of oscillations of potential energy. The graph of U versus x is shown in Fig. 14.2.4

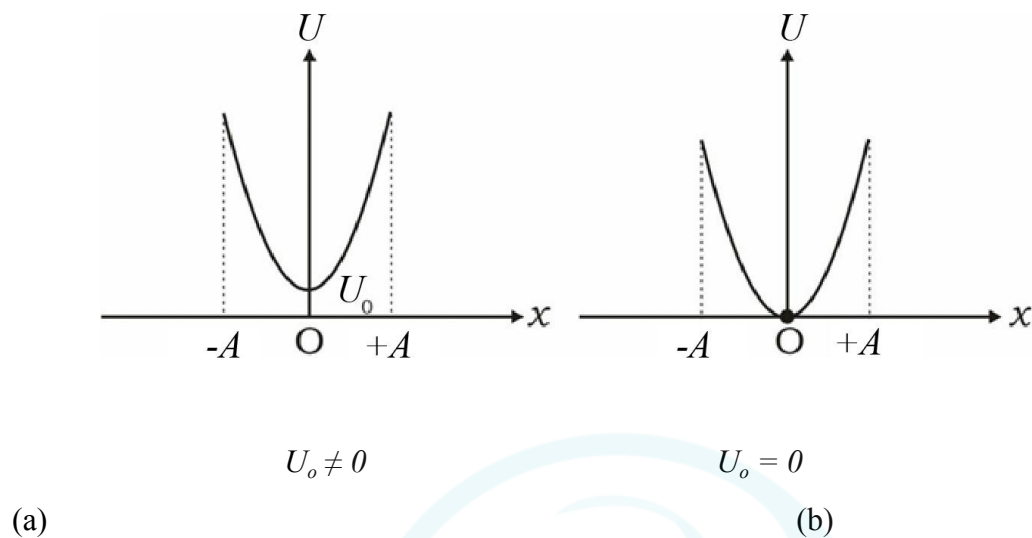


Fig. 14.2.4

The graph is a symmetrical parabola about O.

The total energy (E) of SHM is

$$\begin{aligned}
 E &= K + U \\
 &= \frac{1}{2} m A^2 \omega^2 [\cos^2(\omega t + \phi_o) + \sin^2(\omega t + \phi_o)] \\
 &= \frac{1}{2} m A^2 \omega^2 = \text{a constant} \quad (14.2.11)
 \end{aligned}$$

For $x = A \sin \omega t$; the variation of Displacement and K , U and E versus time (t) is shown in Fig.14.2.5

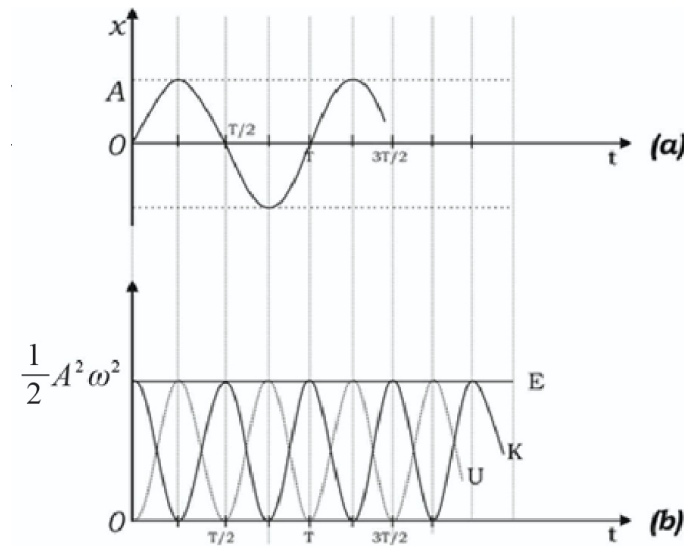


Fig. 14.2.5

From Fig. 14.2.5 (a) and 14.2.5 (b), we have

Time period of SHM = T

Time period of kinetic energy or potential energy = T_k (or T_u) = $\frac{T}{2}$

$$\therefore \frac{T_k}{T_u} = 1 ; \quad \frac{T_k}{T} = \frac{1}{2} = \frac{T_u}{T}$$

In terms of frequencies

$\nu = \frac{1}{T}$ = Frequency of SHM

$\nu_k = \frac{1}{T_k}$ = Frequency of kinetic energy variations

$\nu_u = \frac{1}{T_u}$ = Frequency of potential energy variations

$$\frac{\nu_k}{\nu} = \frac{T}{T_k} = 2 ; \quad \frac{\nu_u}{\nu} = 2 \quad \text{and} \quad \frac{\nu_k}{\nu_u} = 1$$

$\bar{K}$ = Average kinetic energy over one complete cycle

$$= \frac{O+K_m}{2} = \frac{1}{4}mA^2\omega^2 = \frac{E}{2}$$

$\bar{U}$ = Average potential energy over one complete cycle

$$\frac{O+U_m}{2} = \frac{1}{4}mA^2\omega^2 = \frac{E}{2}$$

In SHM, on an average half energy is kinetic and half energy is potential. Also

$$K_m = U_m = E = \frac{1}{2}mA^2\omega^2.$$

Example 14.2.1

A particle of mass 0.2 kg is executing SHM of amplitude 10 cm and angular frequency 50 rad s⁻¹. The particle has an initial phase of $\frac{\pi}{3}$.

- Write an equation describing the SHM.
- Obtain equation for the particle acceleration and show that it differs in phase with displacement by π .
- Calculate the ratio of instantaneous kinetic energy and potential energy for $x = 2.5$ cm.
- For what value of displacement the kinetic energy and potential energy are equal?

Solution:

$$m = 0.2 \text{ kg}, A = 10 \text{ cm} = 0.1 \text{ m}, \omega = 50 \text{ radian s}^{-1}, x = 2.5 \text{ cm} = (1/40) \text{ m}, \phi_o = \frac{\pi}{3}$$

- The standard equation for the SHM is

$$x = A \sin(\omega t + \phi_o)$$

Substituting values of A , ω and ϕ_o , we get for displacement (in m) and time (in sec),

$$x = (0.1) \sin(50t + \frac{\pi}{3}) \quad (14.2.12)$$

- Differentiating Eq(14.2.12) w.r.t. time, we have

$$V = \frac{dx}{dt} = 50(0.1) \cos(50t + \frac{\pi}{3})$$

$$= 5 \cos(50t + \frac{\pi}{3})$$

$$\text{Acceleration, } a = \frac{dV}{dt}$$

$$= -250 \sin(50t + \frac{\pi}{3})$$

$$= 250 \sin(50t + \frac{\pi}{3} - \pi) \quad (14.2.13)$$

$$= 250 \sin(50t - \frac{2}{3}\pi) \quad (14.2.14)$$

From Eqs(14.2.14) and (14.2.11); the phase difference between 'a' and 'x' is

$$(50t - \frac{2}{3}\pi) - (50t + \frac{\pi}{3}) = -\pi \quad (14.2.15)$$

We have

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 \quad \text{For } x = 2.5 \text{ cm}$$

$$PE = \frac{1}{2} (0.2)(50)^2 (\frac{2.5}{100})^2$$

$$= \frac{1}{2} \times \frac{0.2 \times 50 \times 50 \times 2.5 \times 2.5}{100 \times 100} = \frac{5}{32} \text{ J}$$

Also

$$KE = \frac{1}{2}m\omega^2 (A^2 - x^2)$$

$$= \frac{1}{2} \times (0.2)(50)^2 \left[\left(\frac{1}{10} \right)^2 - \left(\frac{1}{40} \right)^2 \right] = \frac{75}{32} \text{ J}$$

$$\therefore \frac{KE}{PE} = \frac{75/32}{5/32} = 15:1$$

iii) Let x be the displacement, when $KE = PE$. Therefore

$$\frac{1}{2} m \omega^2 (A^2 - x^2) = \frac{1}{2} m \omega^2 x^2$$

$$\text{or } 2x^2 = A^2$$

$$\therefore x = \frac{A}{\sqrt{2}} = \frac{10}{\sqrt{2}} \text{ cm} = 5\sqrt{2} \text{ cm}$$

Example 14.2.2

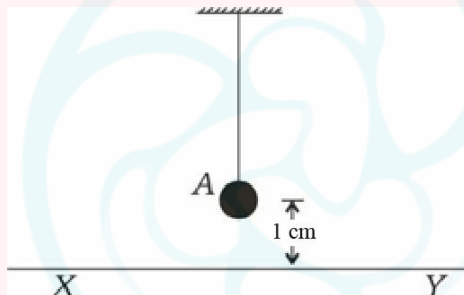


Fig. 14.2.6

A second's pendulum has a bob of mass 10 g. It oscillates with an amplitude of 4 cm, as shown in Fig 14.2.6. What is

- its minimum potential energy
- its maximum kinetic energy
- the total energy pertaining to the SHM
- its maximum potential energy

Take XY as zero of gravitational potential energy. Given $g = 10 \text{ ms}^{-2}$; $\pi^2 \approx 10$

Solution:

Note that the minimum potential energy in this case is not zero. It is the gravitational potential energy at the mean position, i.e. when the bob is in its mean position (A).

We have $m = 100 \text{ gm} = 0.1 \text{ kg}$; $h_{\min} = 1 \text{ cm} = 10^{-2} \text{ m}$

i) $U_{\min} = mgh_{\min} = 0.1 \times 10 \times 0.01 = 0.01 \text{ J} = 10 \text{ mJ}$

ii) For second's pendulum $T = 2 \text{ s}$; therefore

$$\omega = \frac{2\pi}{T} = \pi \text{ rad s}^{-1}$$

$$\begin{aligned} \text{Maximum kinetic energy} &= \frac{1}{2} m \omega^2 A^2 = \frac{1}{2} (0.1) (\pi^2) \left(\frac{4}{100}\right)^2 \\ &= 8 \times 10^{-4} \text{ J} = 0.8 \text{ mJ} \end{aligned}$$

iii) Total energy of SHM = Maximum kinetic energy
= 0.8 mJ

iv) Maximum potential energy = Total energy of SHM. + $U_{\min}$
= 0.8 + 10 = 10.8 mJ



Let Us Reflect

Q.14.2.4 Obtain an expression of KE of the simple harmonic oscillator of Sec 14.2.3 as a function of displacement x and draw the graph of KE vs x . Superpose the graph (as dotted line) on the PE vs x graph of fig 14.2.4(b) and mark the position corresponding to equality of PE and KE ,



Answers

A.14.2.4 With symbols having their usual meanings, we have

$$KE = \frac{1}{2} mV^2 = \frac{1}{2} m\omega^2(A^2 - x^2)$$

The KE vs x graph is a parabola, which has a negative curvature, and PE vs displacement graph has a positive curvature (Fig. 14.2.7).

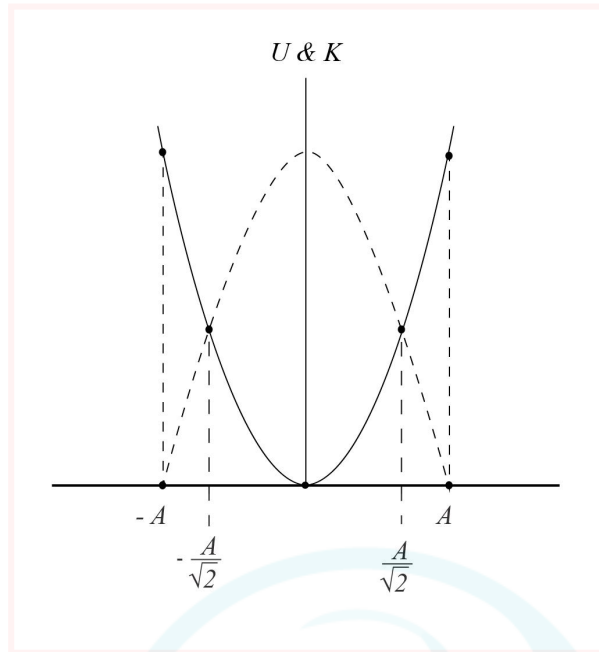


Fig. 14.2.7

The point $x = 0$ will correspond to maximum $KE = \frac{1}{2} m \omega^2 A^2$, which is also the constant value of total energy

$$KE = PE \text{ for } x^2 = A^2 - x^2$$

$$\text{or, } 2x^2 = A^2, \text{ i.e. } x^2 = \frac{A^2}{2}$$

$$\text{or, } x = \pm \frac{A}{\sqrt{2}} .$$

14.2.4 Summary

- For an SHM described by

$$x = A \sin(\omega t + \phi_o) \quad (14.2.13)$$

- (1) Instantaneous velocity, V , is

$$V = A\omega \cos(\omega t + \phi_o) = V_o \cos(\omega t + \phi_o)$$

where $V_o = A\omega$ = maximum velocity. V_o is referred to as velocity amplitude. The relation between V and x is

$$V = \omega (A^2 - x^2)^{1/2}$$

- (2) The instantaneous acceleration, a , is

$$a = -A\omega^2 \sin(\omega t + \phi_o) = -a_o \sin(\omega t + \phi_o) = -\omega^2 x$$

$a_o = A\omega^2$; is the amplitude of acceleration.

- Displacement(x) and instantaneous velocity (V) are not in phase with one another. Also, velocity leads to acceleration by $\frac{3\pi}{2}$ (or $\frac{3T}{4}$).
- The displacement (x) and the instantaneous acceleration (a) are always out of phase with one another.
- The instantaneous kinetic energy (K) of a particle of mass m executing SHM described by Eq (14.2.13) is

$$K = \frac{1}{2} m A^2 \omega^2 \cos^2(\omega t + \phi_o) = K_m \cos^2(\omega t + \phi_o)$$

where $K_m = \frac{1}{2} m A^2 \omega^2$ = Maximum kinetic energy of SHM.

The instantaneous potential energy (U) of a particle of mass (m) executing SHM

described by Eq (14.2.13) is

$$U = \frac{1}{2}kx^2 = \frac{1}{2}mA^2\omega^2 \sin^2(\omega t + \phi_0)$$

$$= U_m \sin^2(\omega t + \phi_0)$$

where $U_m = \frac{1}{2}mA^2\omega^2$ = Maximum potential energy of SHM. The above

expressions for U corresponds to zero potential energy in the mean position.

- Kinetic energy (K) and potential energy (U) vary with time(t) or displacement(x). The total energy of SHM; $E = K + U = \text{constant}$.
- For SHM of time period, T ; the time period of K (or U) is $T/2$; therefore

$$\frac{\text{Time period of S.H.M}}{\text{Time period of K.E. (or P.E)}} = 2$$

- In terms of Frequency,

$$\frac{\text{Frequency of S.H.M}}{\text{Frequency of K.E (or P.E)}} = \frac{1}{2}$$

- The maximum KE (K_m), the maximum PE (U_m) and the total energy(E) equals one another, i.e,

$$K_m = U_m = E = \frac{1}{2}mA^2\omega^2$$

- $\bar{K}$ = The average KE over one cycle and
 $\bar{U}$ = The average PE over one cycle

$$\bar{K} = \bar{U} = \frac{E}{2} = \frac{1}{4}mA^2\omega^2$$

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