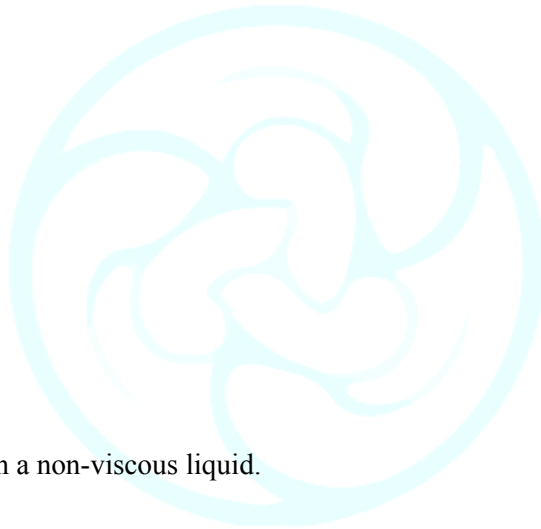


Module 3 Examples of Simple Harmonic Motion

This module is designed to help you to:

- compare vertical and horizontal oscillations of spring mass arrangement;
- analyze combination of springs
 - (i) in parallel
 - (ii) in series and
 - (iii) in mixed grouping;
- describe motion of
 - (i) a simple pendulum.
 - (ii) liquid in a vertical U-tube.
 - (iii) a cylinder floating vertically in a non-viscous liquid.



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14.3.6 Liquid in a U-tube

14.3.7 Summary

14.3.1 Introduction

In the previous modules we were introduced to the concept of SHM and we studied the variation of displacement, velocity, acceleration and energy of particles executing SHM. We also learnt that the energy of a particle or a body executing SHM alternates between potential and kinetic forms, and the motion is supposed to continue forever in the absence of dissipative forces.

In this module we shall study several examples of SHM like oscillations of a mass attached to a spring, motion executed by vertical and horizontal combination of springs and oscillations of a liquid column in a U-tube.

14.3.2 Oscillations of a mass attached to a spring.

An elastic spring is a spring made of an elastic material. For the spring, stretched within its elastic limit, the magnitude of the external force (F) required to produce a change (extensional or compressional) Δl in length of spring is directly proportional to Δl . Expressed mathematically

$$F \propto \Delta l \quad \text{or} \quad F = k (\Delta l)$$

where k is a constant for a given spring known as the spring constant. The spring constant depends on:

- (i) Young's modulus of the material of the spring (Y)
- (ii) area of cross-section (α) of the wire making the spring
- (iii) radius or area of cross-section of spring
- (iv) length of the spring (l).

It can be shown, keeping all other parameters constant that increasing the length of spring decreases k .

Case I: Mass attached to a horizontal spring

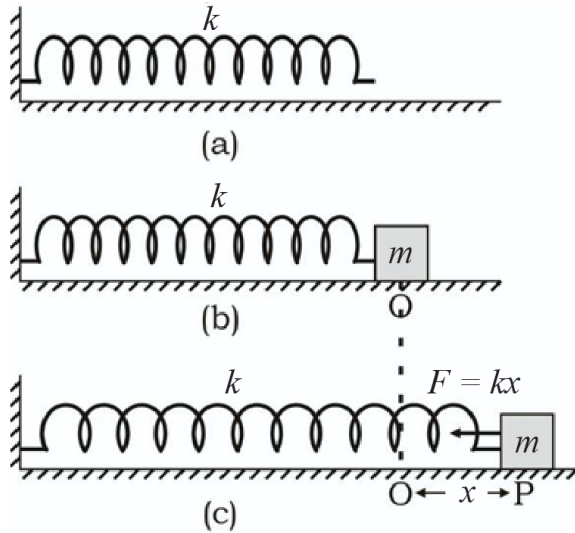


Fig. 14.3.1

Consider a body of mass m lying on a smooth horizontal surface attached to a spring of spring constant k , of natural length l_0 as shown in Fig. 14.3.1 (b). The instantaneous displacement $OP = x$ is shown in

Fig.14.3.1 (c). The force (F) on mass m due to spring has a magnitude kx and is directed towards the mean position O as shown in Fig.14.3.1(c). This is the restoring force which brings mass (m) towards equilibrium (or mean) position O . Expressed mathematically

$$F = -kx$$

Negative sign indicates the restoring nature of the spring force. The equation of motion of mass (m) is

$$m = \frac{d^2x}{dt^2} = -kx$$

$$\text{or } \frac{d^2x}{dt^2} = -\omega^2 x \quad \text{where } \omega^2 = \frac{k}{m}$$

This is the standard equation of an SHM. The mass m executes SHM of time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Here we may note that the time period T does not depend on the amplitude (A) of oscillations.

Case II: Mass attached to a Vertical Spring

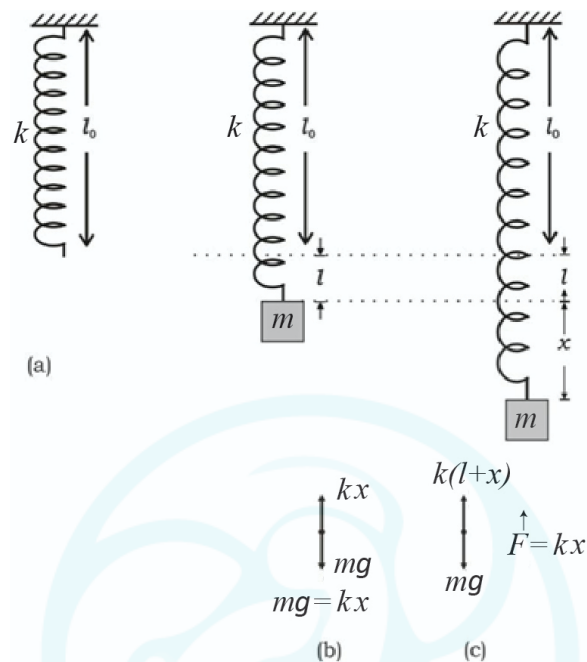


Fig. 14.3.2

Fig. 14.3.2 (a) shows a massless spring of natural length l_0 , hanging vertically from a rigid support.

A mass m attached to the free end of a spring extends it by an amount l . This is the equilibrium position of the system as shown in fig.14.3.2 (b). The forces acting are:

- (i) mg ; vertically downwards
- (ii) restoring force kl ; vertically upwards.

In equilibrium,

$$mg = kl \quad (14.3.1)$$

Fig.14.3.2 (c) shows the instantaneous position (P) of mass (m), when displaced from equilibrium position (O). The instantaneous displacement = $OP = x$

The forces acting on mass (m) are shown in fig.14.3.2 (c). The magnitude of net instantaneous force (F) is

$$F = k(l + x) - mg = kx$$

F is directed towards the mean position O; i.e. it is a restoring force. Obviously m executes SHM of time period (T) given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \quad (14.3.2)$$

From Eq(14.3.1) and Eq(14.3.2), T can be rewritten as

$$T = 2\pi\sqrt{\frac{l}{g}} \quad (14.3.3)$$



Let Us Reflect

Q.14.3.1 Write down the expression of total energy of a spring-mass system (with symbols having their usual meaning). Assuming that the total mechanical energy of the system is conserved, show that the system executes SHM. obtain the time period of the SHM.



Answers

A.14.3.1

Total energy = $E = KE + PE$

$$\therefore E = \frac{1}{2}mV^2 + \frac{1}{2}kx^2$$

From the given condition $\frac{dE}{dx} = 0$

$$\therefore \frac{1}{2}m \frac{d}{dx}(v^2) + \frac{1}{2}k2x = 0$$

$$\text{or } \frac{1}{2}m \cdot 2v \frac{dv}{dx} + kx = 0$$

$$\text{the first term} = mv \frac{dv}{dx}$$

$$= m \frac{dx}{dt} \frac{dv}{dx} = m \frac{dv}{dt}$$

$$= m \frac{d}{dt} \left(\frac{dx}{dt} \right) = m \frac{d^2x}{dt^2}$$

$$\text{So, } m \frac{d^2x}{dt^2} + kx = 0$$

$$\therefore \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Eq(14.3.3) is the differential equation of SHM. So, the system executes SHM. its angular frequency = $\sqrt{\frac{k}{m}}$

$$\therefore \text{time period} = T = 2\pi\sqrt{\frac{m}{k}} \quad (14.3.4)$$

14.3.3 Combination of springs in parallel

Case (i): Vertical arrangement

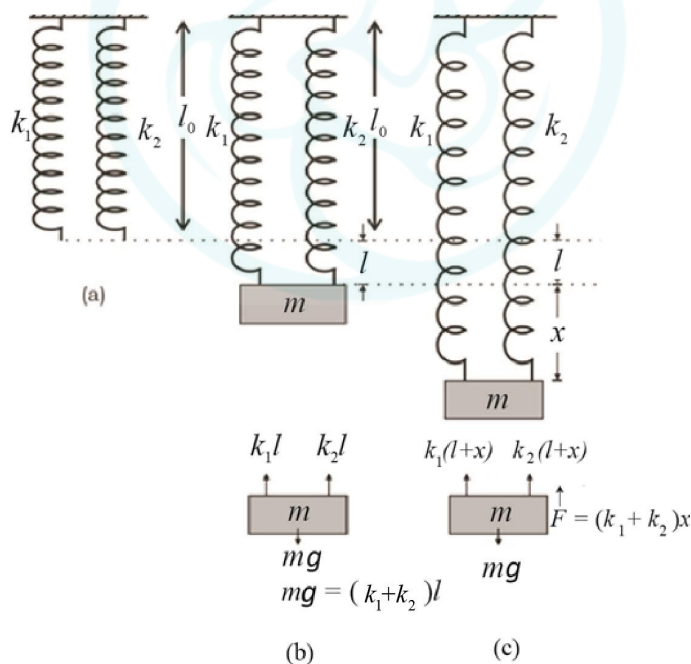


Fig. 14.3.3

Consider a body of mass (m) suspended from the free end of the two springs. In equilibrium position, as shown in fig.14.3.3 (b), (m) is horizontal. The forces acting on mass (m) are shown in Fig 14.3.3(b). In equilibrium,

$$mg = (k_1 + k_2)l$$

Fig.14.3.3 (c) shows mass (m) having an instantaneous displacement $OP = x$. The forces acting on (m) are shown in fig.14.3.3 (c). The net force (F) on mass (m) is

$$\begin{aligned} F &= [k_1(l + x) + k_2(l + x)] - mg \\ &= [k_1(l + x) + k_2(l + x)] - (k_1 + k_2)l = (k_1 + k_2)x \end{aligned}$$

F is directed towards the mean position O .

The equivalent spring constant (k_{eq}) of the arrangement, by definition is

$$k_{eq} = \frac{F}{x}$$

$$\therefore k_{eq} = k_1 + k_2$$

The time period (T) of oscillations of mass (m) is

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{m}{(k_1 + k_2)}}$$

Case (ii): Horizontal Arrangement

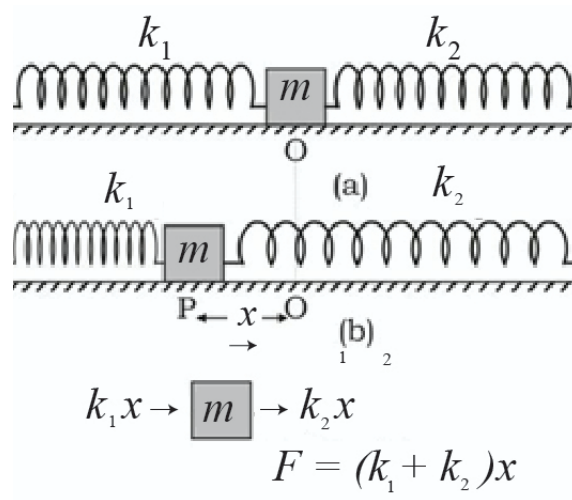


Fig. 14.3.4

Fig.14.3.4 (a) shows the equilibrium position of the arrangement. Fig.14.3.4 (b) shows mass m having an instantaneous displacement $OP = x$. Spring k_1 is compressed and spring k_2 is extended by an amount x . The force on mass (m) due to each spring is shown in fig. 14.3.4 (c). The net force

$$F = (k_1 + k_2) x$$

F is directed towards the mean position O i.e, it is a restoring force. Therefore, mass m is executing an SHM.

k_{eq} be the equivalent spring constant of the arrangement i.e.

$$k_{eq} = \frac{F}{x} = k_1 + k_2$$

The time period of oscillations of mass (m)

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{m}{(k_1+k_2)}} \quad (14.3.5)$$

Note: T is same in vertical as well as horizontal arrangement.

14.3.4 Combination of springs in series

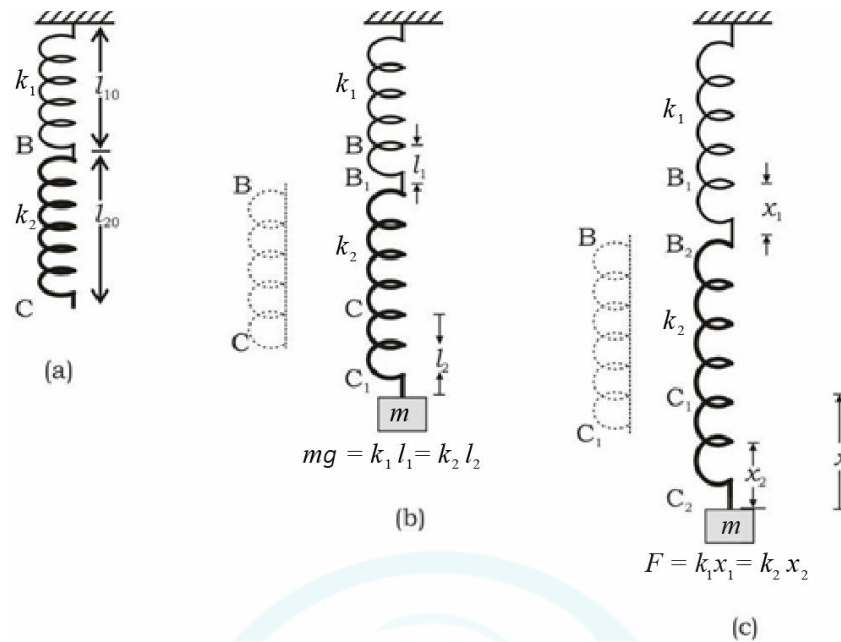


Fig. 14.3.5 (a) two massless springs of spring constants k_1 and k_2 suspended vertically from a rigid support. This arrangement is the arrangement of springs in series. Fig. 14.3.5(b) shows the equilibrium position of the arrangement when a mass m is attached to the free end of spring of spring constant k_2 . Let $BB_1 = l_1 = CC_2 = l_2$ the extension of springs of spring constants k_1 and k_2 respectively.

In equilibrium,

$$mg = k_1 l_1 = k_2 l_2$$

(The forces in the different parts of the arrangement have to be same; for equilibrium)

Fig.14.3.5 (c) shows mass m having an instantaneous displacement $OP = x$.

Let $B_1 + B_2 = x_1$ and $C_1 + C_2 = x_2$ be the instantaneous displacement of springs k_1 and k_2 from their respective equilibrium positions B_1 and C_1 as shown in Fig. 14.3.5 (c.) Obviously,

$$x = x_1 + x_2 \quad (14.3.6)$$

Let F be the additional force measured from the respective equilibrium positions of the two springs. There is no relative motion between different parts of the arrangement therefore

$$F = k_1 x_1 = k_2 x_2 \quad (14.3.7)$$

Let k_{eq} be the equivalent spring constant of the arrangement. By definition the equivalent spring constant, k_{eq} is

$$F = k_{eq} x$$

From Eqs(14.3.6) and (14.3.7) ; we have

$$\frac{F}{k_{eq}} = \frac{F}{k_1} + \frac{F}{k_2}$$

or
$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

The time period (T) of oscillations of mass m is

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{m\left(\frac{1}{k_1} + \frac{1}{k_2}\right)}$$

Example: 14.3.1

A mass of 1.6 kg attached to a spring S_1 executes SHM of time period π seconds. The same mass attached to another spring S_2 executes SHM of time period 2π seconds. What will be the period of oscillation, of the same mass when attached with the two springs joined together as shown in the four cases in **Fig. 14.3.6 (a), (b), (c) and (d)**?

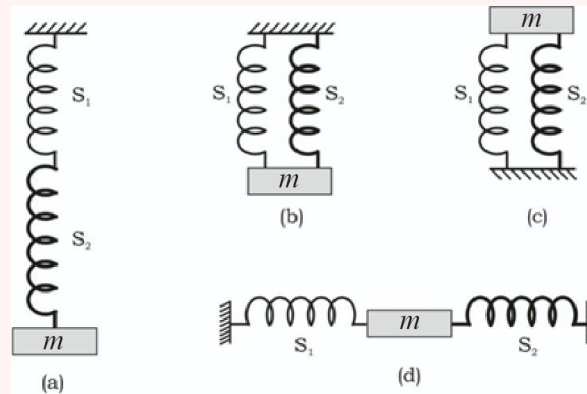


Fig. 14.3.6

Solution

Given $m = 1.6 \text{ kg}$, $T_1 = \pi$. Let k_1 be the spring constant of spring S_1 . Therefore,

$$2\pi\sqrt{\frac{m}{k_1}} = T_1$$

$$2\pi\sqrt{\frac{1.6}{k_1}} = \pi$$

or $\therefore k_1 = 6.4 \text{ Nm}^{-1}$

Given for spring S_2 : $T_2 = 2\pi$, Therefore

$$2\pi = 2\pi\sqrt{\frac{1.6}{k_2}}$$

or $k_2 = 1.6 \text{ Nm}^{-1}$

In fig.14.3.6 (a); the springs are connected in series. Therefore

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2} = \frac{6.4 \times 1.6}{6.4 + 1.6} = 1.28 \text{ Nm}^{-1}$$

The time period, T , of oscillations of mass m is

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{1.6}{1.28}} \approx 2.2\pi \text{ s}$$

In Fig.14.3.6 (b), (c) and (d) ; the springs are joined in parallel. Therefore, in all the three cases

$$k_{eq} = k_1 + k_2 = 6.4 + 1.6 = 8.0 \text{ Nm}^{-1}$$

The time period, T , of oscillations of mass m is

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{1.6}{8.0}} = 2\pi\sqrt{0.2} \approx 0.88\pi \text{ s}$$

14.3.5 Simple pendulum

A simple pendulum is a small-sized ball (regarded as a point mass) attached to a massless; inextensible string. It is suspended from a rigid support. The length of the pendulum (l) is the distance between the point of suspension A and C.G of bob (O).

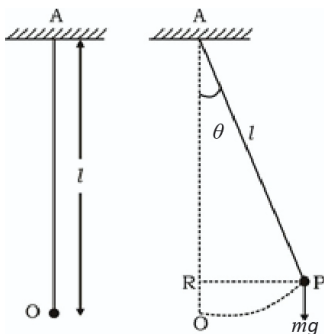


Fig.14.3.7 (a) shows the equilibrium position of the pendulum in a vertical plane. Fig.14.3.7 (b) shows a pendulum having an angular displacement θ . The bob of the pendulum moves along the arc of the circle of radius l , executing angular

oscillations. The external force on the bob is equal to its weight in mg . There is restoring torque (τ) due to mg about the point of suspension A. The instantaneous torque (τ) is

$$\tau = mg \times RP = mg \times AP \sin\theta = mgl \sin\theta$$

The equation of rotational motion of the pendulum is

$$I\alpha = \tau$$

$$\text{or } I \frac{d^2\theta}{dt^2} = - mgl \sin\theta \quad (14.3.8)$$

where I is the moment of inertia of the bob of the pendulum about a horizontal axis through the point of suspension A. In general, Eq(14.3.8) does not represent SHM. However for small amplitude oscillations, $\sin\theta = \theta$, Eq(14.3.8) reduces to

$$I \frac{d^2\theta}{dt^2} = - (mgl\theta)$$

$$\text{Or, } \frac{d^2\theta}{dt^2} = - (\omega^2\theta) \quad (14.3.9)$$

$$\text{where } \omega^2 = \frac{mgl}{I}$$

Eq(14.3.9) is the standard equation of SHM. The time-period (T) of oscillations of the pendulum is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}}$$

Regarding bob as a point mass; $I = ml^2$

Therefore,

$$T = 2\pi\sqrt{\frac{l}{g}} \quad (14.3.10)$$

Note: T does not depend on (i) amplitude of oscillations, and (ii) mass of bob.



Let Us Reflect

Q.14.3.2 For arriving at Eq(14.3.10), we have used the approximation $\sin\theta = \theta$, for θ being very small. Make an assessment of the approximation using $\theta = 18^\circ$, by obtaining the values of θ and $\sin\theta$ correct upto 3 decimal places and find by what percent is θ greater than $\sin\theta$ (Take $\sqrt{5} = 2.236$, and $\pi = 3.14$)

$$\text{A.14.3.2 } \theta = 18^\circ = \frac{180^\circ}{10} = \frac{\pi}{10} = \frac{3.14}{10} = 0.314$$

$$\sin\theta = \sin 18^\circ = \frac{\sqrt{5}-1}{4} = \frac{2.236-1}{4}$$

$$\therefore \sin\theta = \frac{1.236-1}{4} = 0.309$$

the required percentage difference

$$= \frac{0.314 - 0.309}{0.314} \times 100$$

$$= \frac{500}{314} = 1.59 \approx 1.6$$

So θ is 1.6% above $\sin\theta$, for $\theta = 18^\circ$

(Note: There is a misgiving among the students, that in order to make the approximation required for Eq (14.3.10) to hold good, θ should be within 4° . The above analysis shows that such an argument is not sacrosanct. The degree of validity of $\sin\theta \approx \theta$ increases as $\theta \rightarrow 0$. Thus the motion of a simple pendulum being simple harmonic is always an approximation, and it holds with more and more assurance as $\theta \rightarrow 0$)

14.3.6 Liquid in a U-tube

Consider a U-tube with a liquid column of length L in equilibrium between points P and Q [Fig. 14.3.8(a)]. Let ρ denote the density of the liquid. Displace the liquid in the left limb downwards through $PP_1 = y$ [Fig 14.3.8(b)]. The liquid rises in

the right limb to Q_2 . The weight of liquid between P_1 and Q_1 balances one another. The net restoring force F is due to the weight of the liquid column of vertical height $Q_1 Q_2 = 2y$.

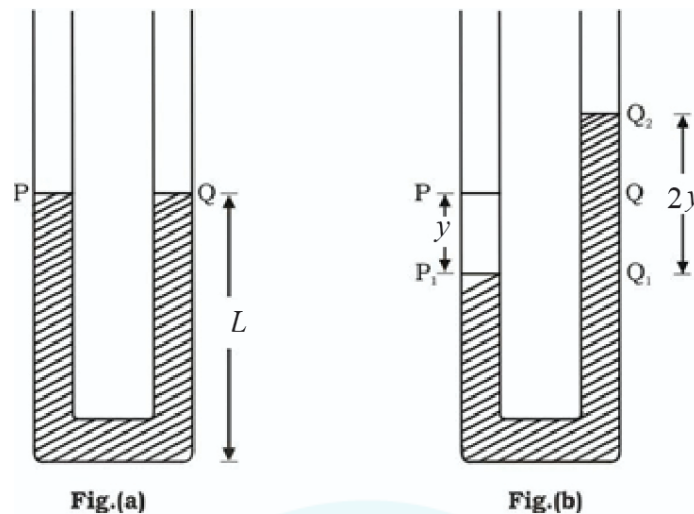


Fig. 14.3.8

$F = \text{Weight of liquid between } Q_1 Q_2 = 2y A \rho g$

where A is the area of cross-section of each limb of the U-tube. F is directly proportional to y , and is directed towards the mean (or equilibrium) position (Q). Therefore, the liquid column executes SHM.

Let m be the total mass of liquid in the U-tube. The equation of motion of liquid is

$$m \frac{d^2 y}{dt^2} = - (2A \rho g) y$$

$$\text{or } \frac{d^2 y}{dt^2} = - \omega^2 y$$

$$\text{where } \omega^2 = \frac{2A \rho g}{m}$$

The time period (T) of oscillations of the liquid column is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{2A\rho g}}$$

The mass, m of the liquid column is $m = 2AL\rho$. Therefore,

$$T = 2\pi\sqrt{\frac{L}{g}} \quad (14.3.11)$$

Note: T does not depend on

- (i) area of cross section (A) of each limb of U-tube, and
- (ii) density (ρ) of liquid inside U-tube.

14.3.7 Oscillation of a vertical cylinder floating in a liquid

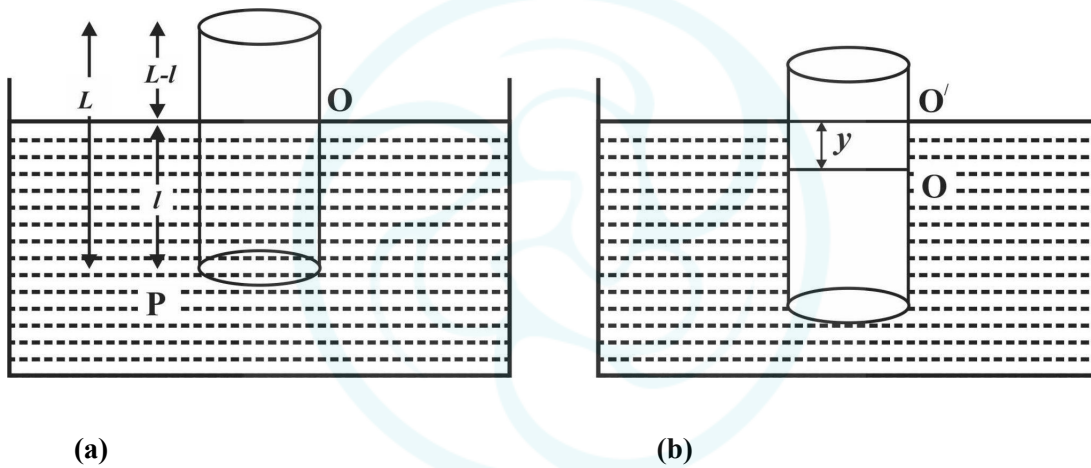


Fig. 14.3.9

Fig.14.3.9(a) shows a cylinder of length L , mass m , area of cross-section A and density σ floating vertically in a liquid of density ρ ($\rho > \sigma$). In equilibrium. l' denotes the length of the cylinder submerged inside liquid. We have

weight of cylinder = weight of liquid displaced

$$\text{or } AL\sigma g = Al\rho g ; \quad L\sigma = l\rho \quad (14.3.12)$$

If the cylinder is pushed downwards through a small distance $OP = y$ as shown in Fig 14.3.9(b). There is an increase in the buoyant force due to the additional liquid displaced by the cylinder. A net upward force F acts on the cylinder. This force

(F) tends to bring the cylinder back to its equilibrium position i.e. it is restoring force. Obviously, F equal weight of the additional liquid displaced. Therefore

$$F = Ay\rho g$$

The equation of motion of cylinder is

$$\frac{d^2y}{dt^2} = - \left(\frac{A\rho g}{m} \right) y$$

$$\text{or } \frac{d^2y}{dt^2} = - \omega^2 y \quad \text{where } \omega^2 = \left(\frac{A\rho g}{m} \right)$$

This is the standard equation of SHM. The time period, T , of oscillations of the cylinder is,

$$T = 2\pi\sqrt{\frac{m}{A\rho g}} \quad (14.3.13)$$

The mass m of cylinder = $m = AL\sigma$. Substituting value of m in Eq(14.3.13), we have

$$\begin{aligned} T &= 2\pi\sqrt{\frac{AL\sigma}{A\rho g}} \\ &= 2\pi\sqrt{\frac{L\sigma}{\rho g}} = 2\pi\sqrt{\frac{l\rho}{\rho g}} \end{aligned} \quad \text{using Eq(14.3.12)}$$

$$\therefore T = 2\pi\sqrt{\frac{l}{g}}$$

Time period of oscillations is same as that of simple pendulum of length $l = L\sigma/\rho$.



Let Us Reflect

Q.14.3.3 A very small body of mass m is loaded at the midpoint O of the straight line, line AB ($= 2a$) joining two heavy masses, each equal to m as shown in Fig 14.3.10

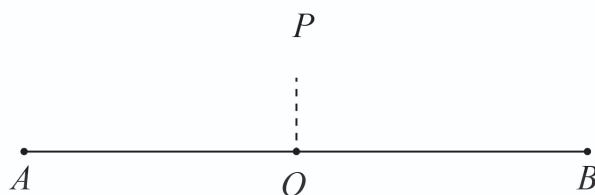


Fig. 14.3.10

The mass m is in equilibrium under the influence of the gravitational attraction of the heavier masses. If it is pulled by a very small distance, x ($x \ll a$) is perpendicular to AB and let go. Show that it executes SHM. Obtain the time period of the SHM.



Answers

A.14.3.3

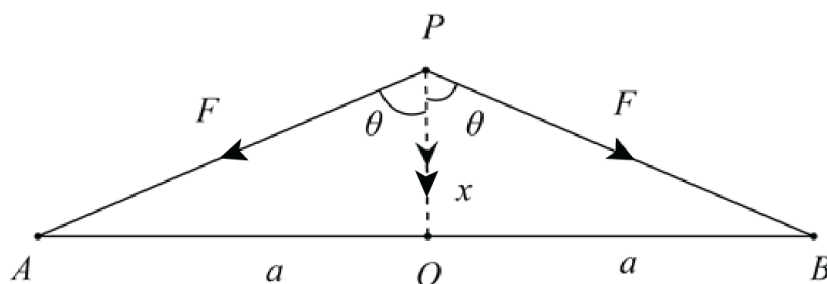


Fig. 14.3.11

Refer to fig 14.3.11 . $AB = 2a$, $OA = OB = a$ The mass m is displayed through
 $OP = x$, where $x \ll a$. the force experienced by m due to M at A

$$= \frac{GMm}{PA^2} \text{ along PA}$$

$$= \frac{GMm}{(a^2 + x^2)} \text{ along PA}$$

$$\text{let, } \frac{GMm}{(a^2 + x^2)} = F$$

Similarly, the force experienced by it due to M at B is also equation $F = \frac{GMm}{(a^2 + x^2)}$

their resultant = $2F \cos\theta$ along PO

$$= \frac{2GMm}{(a^2 + x^2)} \cdot \frac{x}{\sqrt{a^2 + x^2}}$$

$$\frac{2GMm}{(a^2 + x^2)^{\frac{3}{2}}}$$

$$\frac{2GMmx}{a^3 \left(1 + \frac{x^2}{a^2}\right)^{\frac{3}{2}}} = F_0, \text{ say}$$

$$F_0 = \left(\frac{2GMm}{a^3} \right) x, \text{ as } x \ll a$$

So F_0 is proportional to the displacement, x and is opposite to the direction of displacement . so the motion is simple harmonic

$$\text{the force constant} = \frac{GMm}{a^3} = k, \text{ say}$$

$$\therefore \text{The period} = 2\pi \sqrt{\frac{m}{k}}$$

$$= 2\pi \sqrt{\frac{a^3}{GMm}}$$

14.3.7 Summary

- For an elastic spring ; the restoring force, F , due to a compressional or extensional change Δl , in the length of spring is

$$F = k(\Delta l)$$

k is the spring constant of the spring .

- For a given spring, an increase in its original length without any other change decreases the spring constant i.e,

$$k \propto \frac{1}{l_0}$$

- A mass m , attached to the free end of a spring of spring constant k , executes
(i) horizontal or (ii) vertical oscillations, of time period T .

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- For two springs of spring constant k_1 and k_2 connected to one another in
(i) parallel

The equivalent spring constant, k_p is given by

$$k_p = k_1 + k_2$$

(ii) series

The equivalent spring constant k_s is given by

$$\frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2}$$

- A simple pendulum is a point mass, (m); suspended by a massless, inextensible string of length l . The small amplitude oscillations are simple harmonics. The time period of oscillations T , is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

T does not depend on:

- (i) mass of bob
- (ii) amplitude (small) of oscillations
- A liquid in a vertical U-tube executes oscillations of time periods, T .

$$T = 2\pi\sqrt{\frac{m}{2A\rho g}} = 2\pi\sqrt{\frac{L}{g}}$$

m , L and ρ denote the total mass; length of the liquid column in each limb of U-tube and density of liquid respectively. A is the area of cross-section of each limb of the U-tube.

Note: T does not depend on (i) A , and (ii) ρ .

- A cylinder of mass m ; length L density σ executes vertical oscillations about its equilibrium position inside a liquid of density ρ . Let l be the length of the cylinder inside the liquid, in equilibrium. The time period, T , of oscillations is

$$T = 2\pi\sqrt{\frac{m}{2A\rho g}}$$

Also $m = AL\rho g$; therefore

$$T = 2\pi\sqrt{\frac{L\sigma}{\rho g}} = 2\pi\sqrt{\frac{l}{g}}$$

where $l = \frac{L\sigma}{\rho}$

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