

Module 4 : Free, Damped and Forced oscillations

This module is designed to help you to:

- distinguish between free and damped harmonic oscillations;
- derive the equation of damped harmonic motion and explain the meaning of the symbols used therein;
- distinguish between free and forced vibrations;
- identify examples of resonance from real life situations.

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14.4.1 Introduction

So far we have studied about the different features of SHM and these have been substantiated through multiple examples. Largely, we have dealt with ideal situations where dissipation of energy has not been taken into account. But in real life situations we come across cases where oscillation dies out due to the presence of dissipative forces. This type of oscillation is called damped. We shall study about that.

Further, for sustenance of oscillations, the system needs to be driven by an external periodic force and the resulting motion is called forced vibration. So, we shall study the features of damped forced vibrations and a special feature of forced vibration called 'Resonance', which occurs when the frequency of the external periodic motion is equal to the natural frequency of vibration of the oscillation.

14.4.2 Free harmonic oscillations

Harmonic oscillations in the absence of any resistive or dissipative force are known as free harmonic oscillations. The only force acting is the restoring force, $F = -kx$. For an oscillator of mass m , executing SHM along x-axis, the equation of motion is

$$m \frac{d^2x}{dt^2} = -kx \quad (14.4.1)$$

The variation of instantaneous displacement, x , with time is

$$x = A \sin(\omega t + \phi_0) \quad (14.4.2)$$

A = amplitude of motion; ω = angular frequency of SHM = $\sqrt{\frac{k}{m}}$.

ϕ_0 = initial phase of SHM

The characteristics of free harmonic oscillations are:

- 1) Amplitude of motion does not depend on time.
- 2) Oscillations once started never stop.
- 3) There is no loss in energy given to the system initially.
- 4) The frequency of oscillation (ω) is the natural frequency of oscillator.

14.4.3 Damped oscillations

Free harmonic oscillations is an ideal concept. In practice oscillations occur in a medium. The oscillator experiences an external force due to viscosity of the medium. This force is known as the damping force. There is loss in energy of the oscillator in doing work against the resistive or dissipative forces. This causes the amplitude of oscillations to decrease as time increases and ultimately oscillations stop.

In general, expression for damping forces is complicated. For simplicity let us consider a situation where damping force is directly proportional to instantaneous velocity (v) of oscillator, i.e.

$$|F_{damp}| = \lambda v$$

λ is a constant known as damping constant. λ depends on the nature of the medium. For a particle of mass m the equation of damped harmonic oscillations is

$$m \frac{d^2 x}{dt^2} = -kx - \lambda \left(\frac{dx}{dt} \right)$$

or

$$\frac{d^2 x}{dt^2} + 2b \frac{dx}{dt} + \omega^2 x = 0 \quad (14.4.3)$$

where $2b = \frac{\lambda}{m}$. b is also used to represent damping constant. It is half of the damping force per unit mass and unit velocity of the oscillator. It is introduced primarily for mathematical convenience of solving the differential equation. And $\omega^2 = \frac{k}{m}$. ω is the natural frequency of free oscillations. The dimensions of ω and b are both the same as that of reciprocal of time.

It can be shown that the instantaneous displacement (x) is :

$$\begin{aligned} x(t) &= A_0 e^{-bt} \sin(\omega_1 t + \phi_0) \\ &= A \sin(\omega_1 t + \phi_0) \end{aligned} \quad (14.4.4)$$

where A_0 is the amplitude of oscillation at $t = 0$ and ω_1 is the angular frequency of oscillations. The characteristics of damped oscillations are:

- (i) Amplitude of oscillation is not constant. It decreases exponentially with time such that the instantaneous amplitude of oscillations $A(t) = A_0 e^{-bt}$ Fig. 14.4.1 shows damped harmonic oscillations.
- (ii) The rate of decrease of amplitude depends on the damping constant (b) of the medium.
- (iii) Angular frequency of oscillation ω_1 is not equal to the natural frequency of a free harmonic oscillator. It can be shown that

$$\omega_1^2 = \omega^2 - b^2$$

- (iv) Time-period (T) of oscillations is

$$T = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{\omega^2 - b^2}} = \text{constant}$$

However T depends on damping constant (b) of the medium .

- (v) There is a loss in energy of the oscillator. The energy E of the oscillator decreases exponentially with time.

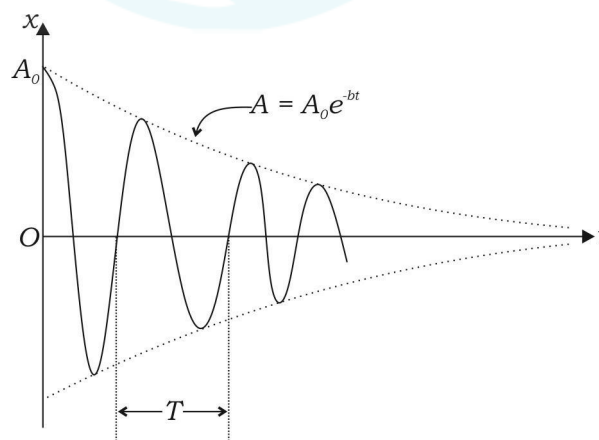


Fig. 14.4.1 damped harmonic oscillations

14.4.4 Forced Oscillations

Oscillations of an oscillator under an external applied periodic force are known as forced harmonic oscillations. Oscillations of constant amplitude are produced by applying an external force (F_{ext}) so as to compensate for the instantaneous loss of energy due to the damping force . The damping force, being

directly proportional to the instantaneous velocity (v) of the oscillator, varies periodically with time. Therefore the external applied force has to be a periodic force. Let

$$F_{ext} = F_0 \sin pt$$

where ; F_0 = Amplitude of applied force and p = Angular frequency of external applied force.

In general $p \neq \omega$.

The equation of motion of a forced harmonic oscillator is

$$m \frac{d^2x}{dt^2} = -kx - \lambda \frac{dx}{dt} + F_0 \sin pt$$

or $\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + \omega^2 x = f_0 \sin pt$ (14.4.5)

where, $2b = \frac{\lambda}{m}$, $\omega^2 = \frac{k}{m}$ and $f_0 = \frac{F_0}{m}$

Initially oscillations of system are not steady because oscillator tries to oscillate with its own natural frequency (ω) and the applied external force tries to drive the oscillator with its frequency (external force). In steady-state, oscillator executes oscillations of (i) constant amplitude (A) and (ii) frequency equal to the frequency of external applied force (p). It can be shown that steady-state oscillations are given by

$$x(t) = A \sin(pt + \phi_0) \quad (14.4.6)$$

$$\text{where } A = \frac{f_0}{[(\omega^2 - p^2)^2 + 4b^2 p^2]^{\frac{1}{2}}}, \quad (14.4.7)$$

$$\text{and } \phi_0 = \tan^{-1} \left(\frac{2bp}{\omega^2 - p^2} \right) \quad (14.4.8)$$

The main characteristics of forced harmonic oscillations are

- 1) Amplitude of oscillations is constant.
- 2) The amplitude of oscillations depends on
 - (a) the frequency of applied external force, p .
 - (b) natural frequency (ω) of free oscillations
 - (c) damping constant, b .



Let Us Reflect

Q.14.4.1 Using Eqs(14.4.2) and (14.4.3) obtain expression for the velocity of the forced oscillation.

Q.14.4.2 Use the expression for the velocity obtained in problem Q.14.4.1, and find the value of p for which velocity is maximum.



Answers

A.14.4.1 $v = \frac{dx}{dt} = Ap \cos(pt + \phi_0)$

$$= \frac{f_0 p \cos(pt + \phi_0)}{\left\{(\omega^2 + p^2)^2 + 4b^2 p^2\right\}^{1/2}}$$

$$= \frac{f_0 \cos(pt + \phi_0)}{\left\{\left(\frac{\omega^2}{p} - p\right)^2 + 4b^2\right\}^{1/2}}$$

A.14.4.2 Velocity is maximum, when the denominator of the expression is minimum.

The Deno = $\left\{\left(\frac{\omega^2}{p} - p\right)^2 + 4b^2\right\}^{1/2}$.

The second term inside the bracket is a constant, and the first term, which is a variable, is a perfect square. The denominator is minimum, when the perfect square is equal to zero.

i.e $\frac{\omega^2}{p} = p$

or $p^2 = \omega^2$

$\therefore p = \omega$ = The natural frequency of vibration of the driven oscillation. It is called the phenomenon of velocity resonance.

14.4.5 Resonance

From equation (14.4.2) the amplitude (A) of oscillations is small if (i) $p \ll \omega$ or (ii) $p \gg \omega$. For a particular value $p = p_r$ of the applied external force the amplitude of oscillations is maximum. The

denominator $D = (\omega^2 - p^2)^2 + 4b^2 p^2$, is minimum at $p = p_r$. It can be shown that

$$p_r^2 = \omega^2 - 2b^2$$

p_r is known as the resonance frequency.

For a weakly damped oscillator, i.e. ($b \ll \omega$) ; $p_r = \omega$

The resonance frequency is nearly the same as the natural frequency of oscillations. At resonance the amplitude (A_{max}) of oscillations is given by

$$A_{max} = \frac{f_0}{2b[1+\omega^2]^{\frac{1}{2}}} \quad (14.4.9)$$

Case I : Small Damping

For a lowly damped harmonic oscillator b is a small number, therefore A_{max} is a very large number. In other words, the amplitude of oscillations is very large at resonance.

Case II: High Damping

For a highly damped oscillator b is a large number, therefore A_{max} has a small value.

Fig.14.4.2 (a) and (b) respectively show the variation of amplitude (A) of steady state oscillations as frequency (p) of external applied force varies for lowly and highly damped oscillators.

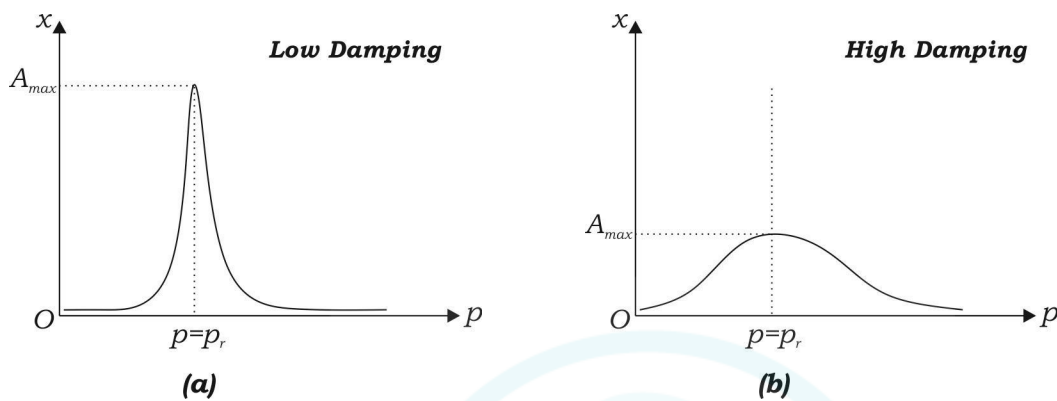


Fig.14.4.2 Resonance

14.4.6 Examples of Resonance

- 1) A "child swing" acquires a large; sustained amplitude of oscillations by synchronising the rhythm of pushing against ground (external periodic force) with the natural frequency of the swing.
- 2) Musical stringed instruments like sitar or guitar make use of the phenomenon of resonance to obtain a stronger and/or more melodious output.
- 3) Resonance can cause dangerous vibrations in a temporary bridge. So, military men do not cross a temporary bridge while marching on it.
- 4) The tuning of a radio (or transistor) uses the phenomenon of resonance. Moving the tuning "knob", varies the natural frequency of the receiver circuit. When this frequency equals the frequency of a desired transmitting station the tuning circuit of radio/transistor executes resonant oscillations of large amplitude. The subsequent circuitry analyses this "signal" and plays that particular station.
- 5) In laboratory, the phenomenon of resonance is used in
 - (a) Sonometer to determine the frequency of a tuning fork.
 - (b) Resonance of air column to determine speed of sound.



Let Us Reflect

Q.14.4.3 While soldiers march along a temporarily constructed bridge in difficult terrains, they are asked to break the rhythm of pulling their steps. Why?



Answers

A.14.4.3 A temporarily constructed bridge is vibrating system generally having a reasonable large frequency. The marching of soldiers above it subjects the bridge to forced vibration, and if the rhythm of marching is such that the condition of resonant vibration gets fulfilled, then there will be a very large amplitude vibration of the bridge which may lead to its collapse. So, the soldiers are asked to break steps, so that the rhythm gets disturbed and the forced vibration, which might have caused serious damage, does not get sustained. It is the reason why resonant vibration, at times, is called sympathetic vibration.

Activity 14.4.1

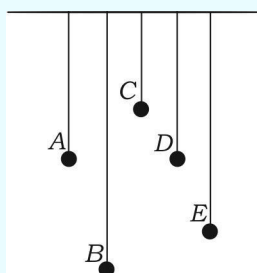


Fig. 14.4.3 pendulums suspended from a common elastic support

Five simple pendulums A, B, C, D, and E are suspended from a common elastic support as shown in Fig. 14.4.3. Length of pendulum A and D is the same. All other pendulums have different lengths.

Displace pendulum A so that it executes SHM. Observe oscillations of B, C, D and E. The amplitude of oscillations of B, C and E is small. However D executes oscillations of large amplitude. Here A acts as a driver; supplying external periodic force. B, C, D, and E are pendulums executing forced oscillations. Since $\nu_A = \nu_D$ resonance occurs, therefore, D executes resonant oscillations of large amplitude.

14.4.7 Summary

- Oscillations under a restoring force only, are the free oscillations. Equation of motion of a free harmonic oscillator is

$$m \frac{d^2 x}{dt^2} = -kx$$

$\omega = \sqrt{\frac{k}{m}}$ is the natural (or free) frequency of oscillations. The oscillations are of constant amplitude and frequency.

- Practically all oscillations are in presence of some resistive or dissipative force. These are known as damped harmonic oscillations. Equation of motion of a damped harmonic oscillator is

$$m \frac{d^2 x}{dt^2} = -kx - \lambda \left(\frac{dx}{dt} \right)$$

λ is damping constant. The amplitude of oscillations decreases as time increases. The instantaneous amplitude of oscillations (t) is

$$A(t) = A_0 e^{-bt}$$

A_0 is amplitude at $t = 0$. $b = \frac{\lambda}{m}$

- To obtain sustained oscillations of constant amplitude; energy is supplied to the oscillator by an external agency so as to exactly compensate for the loss of energy due to the dissipative force. These are forced harmonic oscillations.
- For an external applied force ; $F_{ext} = F_0 \sin pt$, the equation of forced oscillator is

$$m \frac{d^2 x}{dt^2} = -kx - \lambda \left(\frac{dx}{dt} \right) + F_0 \sin pt$$

The steady state oscillations are given by

$$x = A \sin(pt \pm \Phi_0)$$

The amplitude, A of oscillations depends on:

- I. Frequency (p) of external applied force
 - II. Natural frequency (ω) of oscillator
 - III. Damping constant (b).
- At resonance, oscillations of oscillators have a maximum amplitude of oscillations. For a lowly damped oscillator resonance frequency (p_r) is nearly equal to the natural frequency (ω) of an oscillator.

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