

Module 1 Kepler's Laws and Universal Law of Gravitation

This module is designed to help you to:

- analyze the motion of objects in the gravitational field of other objects, orbits of planets and satellites around the central body exerting gravitational force;
- explain Kepler's laws and their applications;
- describe the universal law of gravitation;
- explain the effect of gravitation with the understanding of the relationship between nature and matter on a scientific basis.

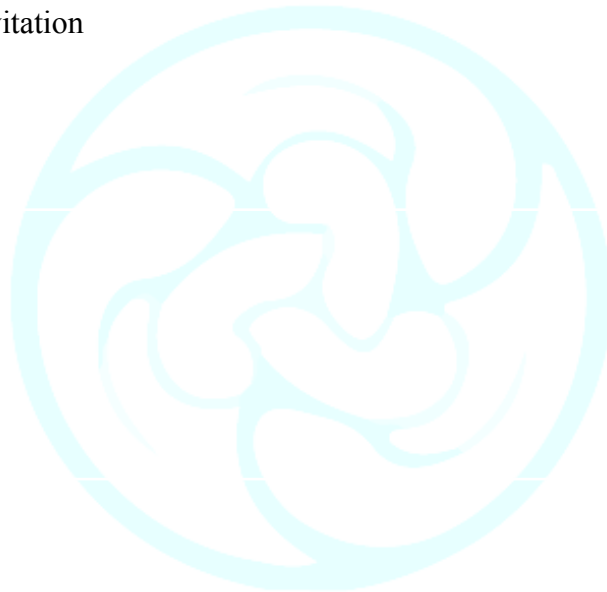
Content

8.1.1 Introduction

8.1.2 Kepler's laws of planetary motion

8.1.3 Universal law of gravitation

8.1.4 Summary



8.1.1 Introduction

We are all aware of the tendency of all material objects to be attracted towards the earth. An object thrown upwards falls down to the earth after attaining a certain height. Observations since early times recognize stars which appeared in the sky with positions unchanged year after year. The more interesting objects are the planets which seem to have regular motions against the background of stars. The planets were described as moving in circles with the center of the circles themselves moving in larger circles. Nicolas Copernicus proposed a model in which planets moved in circles around a fixed central Sun. His theory was discredited by the church, but notable amongst its supporters was Galileo who had to face persecution from the state for his beliefs. Around the same time as Galileo, Tycho Brahe spent his entire lifetime recording observations of planets. His compiled data was analyzed later by Johannes Kepler. He could extract from the data three elegant laws now known as Kepler's laws. These laws were known to Newton and enabled him to propose his Universal law of gravitation.

8.1.2 Kepler's laws

Kepler's three laws of planetary motion are as follows:

First Law: Each planet moves in an elliptical orbit with the Sun at one of its foci.

It is possible to establish this law from direct calculation starting from Newton's law of gravitation.

Refer to Fig.8.1.1.

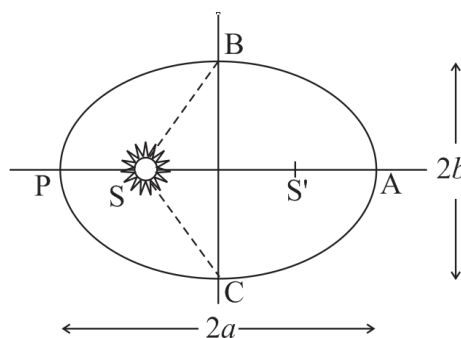


Fig.8.1.1: An ellipse traced out by a planet around the Sun. The closest point is P and the farthest point is A . P is called the perihelion and A the aphelion. The semi-major axis (a) is half the distance AP .

It may be noted that the Sun is at one of the foci, whereas the other focus does not have much physical significance. The existence of two foci is an integral part of the mathematical description of an ellipse.

Earth moves around the Sun in an elliptical orbit. It is a common misconception that seasons on earth are caused due to elliptical orbit. For earth's orbit, eccentricity ($e = 0.016$) is very small and earth's orbit is

nearly circular. Seasons on the earth are caused by tilt of earth's spin axis by about 23.5° with respect to normal to the orbital plane of the earth, which brings different parts of earth closer to the Sun at different times of the year. In december-january the northern hemisphere is away from the Sun causing winter in the northern hemisphere and summer in the southern hemisphere. In june-july it is the other way round causing summer in the northern hemisphere and winter in the southern hemisphere.

Second Law: The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.

This law relates the planet's speed to its distance from the Sun and is a consequence of the law of conservation of angular momentum.

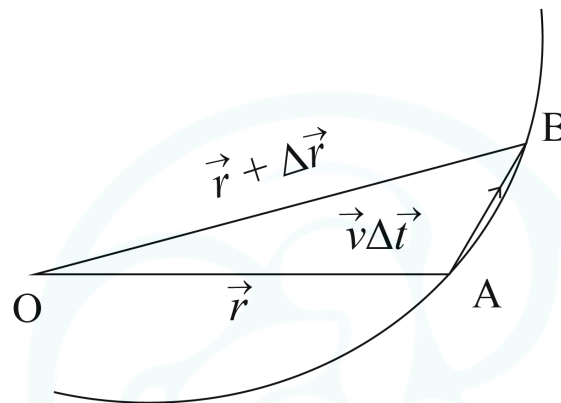


Fig.8.1.2: Radius vector, with reference to the origin O of the planet at the position A is $\vec{r}$. After an infinitesimal time Δt , it is at B , where $\vec{OB} = \vec{r} + \Delta\vec{r}$

Refer to Fig. 8.1.2. Assume the Sun to be at the origin and let the position and momentum of the planet be denoted by $\vec{r}$ and $\vec{p}$ respectively. Let us represent the area swept out by the planet of mass m in the time interval Δt as ΔA .

Then, arc $AB \approx$ chord $AB = \Delta\vec{r}$

$$\text{And, } \Delta\vec{A} = \frac{1}{2} \{ \vec{r} \times (\vec{r} \times \Delta\vec{r}) \}$$

$$= \frac{1}{2} (\vec{r} \times \Delta\vec{r})$$

$$\Delta\vec{A} = \frac{1}{2} (\vec{r} \times \vec{v} \Delta t), \quad \vec{v} \text{ is the velocity of the planet.}$$

$$\frac{\Delta \vec{A}}{\Delta t} = \frac{1}{2}(\vec{r} \times \vec{v})$$

$$\begin{aligned}\frac{\Delta \vec{A}}{\Delta t} &= \frac{1}{2}\left(\vec{r} \times \frac{\vec{p}}{m}\right), \text{ since } \left(\vec{v} = \frac{\vec{p}}{m}\right) \\ &= \frac{\vec{L}}{2m}, \quad \vec{L} \text{ is the angular momentum } (\vec{L} = \vec{r} \times \vec{p} = m.\vec{r} \times \vec{v})\end{aligned}$$

As the gravitational force is radial, there is no external torque. Hence angular momentum is conserved. Smaller r implies larger v .

Since R.H.S. of the equation is constant, $\frac{\Delta \vec{A}}{\Delta t} = \text{constant}$.

This equation describes **Kepler's Law of Areas**.

Let us consider an orbiting planet around the Sun (Fig.8.1.3). At D, the planet has least potential energy and the greatest kinetic energy and at C, it has the greatest potential energy and least kinetic energy. This is due to the fact that total energy remains constant. Therefore at D, the orbital speed of the planet is greatest and it is least at C. Let us suppose that in time dt , the planet at C moves to B with orbital speed v_1 and at position D it moves with orbital speed v_2 from D to E.

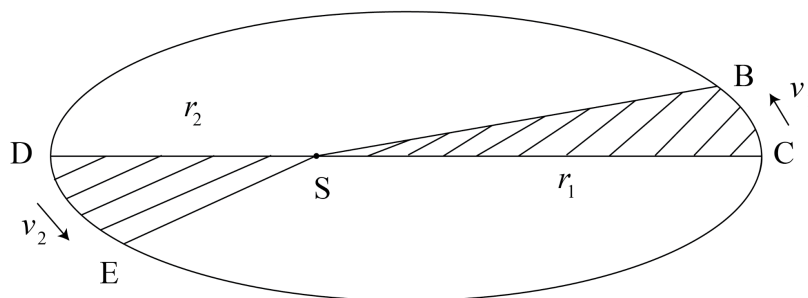


Fig.8.1.3: The shaded area is the area ΔA swept out in a small interval of time Δt .

The area swept out in equal intervals of time, each equal to dt , from C and D are the sectors SCB and SED. But as dt is an infinitesimal, the sectors are almost equal to the areas of the triangles SCB and SED; and CB, DE can be considered perpendicular to SC, SD respectively.

Let A_1 and A_2 be the areas of the triangle BSC and DSE respectively.

We have from Fig. 8.1.3

$$A_1 = \frac{1}{2}r_1v_1dt \quad (4.1.1)$$

$$A_2 = \frac{1}{2}r_2v_2dt \quad (4.1.2)$$

However by Kepler's 2nd law

$$A_1 = A_2$$

$$\therefore \frac{v_1}{v_2} = \frac{r_2}{r_1} \quad (4.1.3)$$

Thus the ratio of the orbital speeds is equal to the inverse of the ratio of distances from the Sun.

For circular orbit

$$r_1 = r_2$$

It implies that $v_1 = v_2 = v$

For a circular orbit the orbital speed of the planet is constant.

Example 8.1.1: Mercury revolves around the Sun in an elliptical orbit. The aphelion of orbit is at 6.98×10^7 km. and its perihelion is at 4.6×10^7 km. Find the ratio of the speeds of Mercury at its aphelion and perihelion.

Solution:

$$\frac{v_1}{v_2} = \frac{r_2}{r_1} = \frac{4.6 \times 10^7}{6.98 \times 10^7} = 0.659.$$

Third Law: The square of the time period of revolution of a planet is proportional to the cube of the semi-major axis of the elliptical orbit of the planet.

This can be established from direct calculations starting from Newton's law of gravitation.

Example 8.1.2: A Saturn year is 29.5 times the earth year. How far is Saturn from the Sun if the Earth is 1.50×10^8 km away from the Sun?

Solution: Both Saturn and Earth revolve around the Sun in elliptical orbits. From Kepler's third law,

$$T = \left(\frac{4\pi^2 r^3}{GM} \right)$$

$$\frac{T_s^2}{T_e^2} = \frac{r_s^3}{r_e^3}$$

$$\begin{aligned}
 r_s &= r_e \left(\frac{T_s}{T_e} \right)^{\frac{2}{3}} \\
 &= 1.5 \times 10^{11} (29.5)^{\frac{2}{3}} \\
 &= 1.43 \times 10^{12} \text{ m}
 \end{aligned}$$



Let Us Reflect

Q.8.1.1 Prove that for a planet moving round the Sun, its orbit is planar.



Answers

A.8.1.1 The force responsible for the motion acts towards the centre. In other words, the angle between the radius vector of the planet, at any position, and the force is 180° . So the vector product of the radius vector and the force is zero, which means that the torque is zero. So the angular momentum of the planet remains conserved.

Now, angular momentum is proportional to the vector product of the radius vector $\vec{r}$ and linear momentum $\vec{p}$ and being the vector product it is perpendicular to $\vec{r}$. So $\vec{r}$ is always perpendicular to a constant vector. So it lies on a plane perpendicular to the constant vector. Hence the orbit of a planet round the Sun is planar.

Activity 8.1.1: Drawing an ellipse- Take a sheet of white paper and fix it on a drawing board. Select two points F_1 and F_2 . Take a string of length L such that $F_1F_2 < L < 2F_1F_2$ and fix its ends at F_1 and F_2 by pins.

With the tip of a pencil stretch the string taut and then draw a curve by moving the pencil keeping the string taut throughout (Fig.8.1.4). The closed curve you get is called an ellipse. Clearly for any point T on the ellipse, the sum of the distances from F_1 and F_2 is a constant (L). F_1 , F_2 are called foci. Join the points F_1 and F_2 and extend the line to intersect the ellipse at points P and A as shown in the figure. The midpoint 'O' of the line PA is the centre of the ellipse and the length $PO = AO$ is called the semi major axis of the ellipse. For a circle the two foci merge into one and the semi-major axis becomes the radius of the circle.

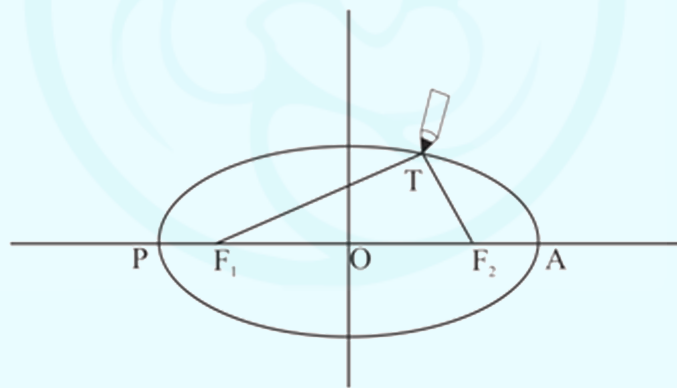


Fig.8.1.4: Drawing an ellipse

This drawing is based on property of ellipse that $r_1 + r_2 = 2a = \text{constant}$, where $F_1T = r_1$, $F_2T = r_2$ and $PO = OA = a$ (length of semi major axis)

Activity 8.1.2 Bode's law (Activity related to Kepler's third law)

This law predicts the mean distance of planets in the solar system in the Astronomical Unit (AU- One astronomical unit is the average distance between the Sun and earth). The law relates the mean distance of the planets from the Sun to a simple mathematical progression of numbers. Beginning with the following simple sequence of numbers.

0, 3, 6, 12, 24, 48, 96, 192, 384....

Adding 4 to each of the numbers, we have

4, 7, 10, 16, 28, 52, 100, 196, 388....

Dividing each of the numbers by 10, we get

0.4, 0.7, 1.0, 1.6, 2.8, 5.2, 10.0, 19.6, 38.8

Given below are the observed distance of the planet from the Sun and time periods of planets in our solar system. Time period in years = (distance in AU)^{3/2}. Calculate the time period for all the planets given in Table 8.1.1 using distance obtained from Bode's law and compare it with the observed ones. Comment on the values obtained in the last column of the Table below, which is based on Kepler's third law and the corresponding equation is

$$T^2 = ka^3$$

Value of k cancels out on finding the ratio $T_{\text{Bode}}/T_{\text{Observed}}$

Table 8.1.1

Planet	Actual distance (a) (in A.U.)	Distance Bode's law I	Observed time period (T) (years)	$T = (k)a^{3/2}$ ('a' from Bode's law) (years) ($k = \text{a constant}$)	$T_{\text{Bode}}/T_{\text{Observed}}$
Mercury	0.39	0.4	0.24		
Venus	0.72	0.7	0.615		
Earth	1.00	1.0	1		
Mars	1.52	1.6	1.88		
Missing Planets (Asteroid Belt)	2.77	2.8			
Jupiter	5.20	5.2	11.9		
Saturn	9.54	10.0	29.5		
Uranus	19.19	19.6	84		

Neptune	30.06	38.8	165		
---------	-------	------	-----	--	--

Comments: It is interesting to note that a missing planet was predicted by Bode's law and now an asteroid belt has been found at the position.

Activity 8.1.3: Students may be facilitated to draw a graph between the square of the time periods of revolution of the planets and the cubes of the semi-major axes of the elliptical orbits of the planets round the Sun.

It will be a straight line passing through the origin.

8.1.3 Universal law of gravitation

It is a common observation that a body thrown or projected from the surface of the earth falls back onto the earth. This is due to the attractive gravitational force of earth on the body. In this module we discuss the gravitational force and the law that govern the force of attraction between two material bodies.

Every object in the universe attracts every other object with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. This force is always attractive.

According to Newton, the magnitude of the force F on a point mass m_2 due to another point mass m_1

$$F = \frac{Gm_1m_2}{r^2} \quad (8.1.4)$$

Where G is a universal gravitational constant and r is the distance between m_1 and m_2 .

This law is valid for point masses. However we can use it for extended bodies if we consider their centre of masses as their respective positions.

In magnitude and direction, the force of attraction $\vec{F}_{21}$ on m_2 due to m_1 ,

$$\vec{F}_{21} = \frac{Gm_1m_2}{r_{21}^2} \vec{r}_{21} \quad (8.1.5)$$

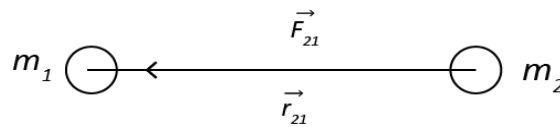


Fig.8.1.5

Where $\hat{r}_{21} = \frac{\vec{r}_{21}}{|\vec{r}_{21}|}$ is a unit vector in the direction of $\vec{r}_{21}$.

$$\vec{F}_{12} = \frac{Gm_1m_2}{r_{12}^2} \hat{r}_{12}$$

Similarly, the gravitational force $\vec{F}_{12}$ on m_1 due to m_2 is

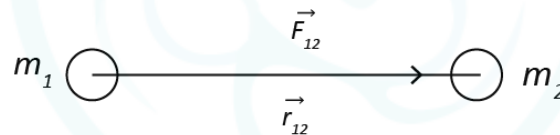


Fig 8.1.6

Where $\hat{r}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|}$ is a unit vector in the direction of $\vec{r}_{12}$.

$$\text{As } \vec{r}_{21} = -\vec{r}_{12}, \quad \vec{F}_{21} = -\vec{F}_{12}$$

Thus gravitational force acting on two bodies is equal in magnitude and opposite in direction.

The gravitational force is a central force.

A central force is a force that points directly towards (or directly away from) a fixed point (centre), and whose magnitude only depends on the distance of the object from the centre. Another example of central forces is Coulomb force between two charges.

For example, in the case of the earth-moon system, force experienced by the moon towards the earth is directed towards the center of mass of the earth which acts as the centre of force. Similarly, force experienced by the earth towards the moon is directed towards the center of mass of the moon which acts as the centre of force.



Let Us Reflect

Q.8.1.2 Is it possible for a body to have inertia but no weight?

Q.8.1.3 How is the gravitational force between two point masses affected when they are dipped in water keeping the separation between them the same?



Answers

A.8.1.2 Mass is the measure of inertia of a body, but weight is the gravitational force experienced by the body. A body will always have mass but the gravitational force on it can be zero, for example, when it is kept at the center of the earth. Hence, the answer is yes.

A.8.1.3 It remains the same as the gravitational force is independent of the medium separating the masses.

Example 8.1.3 Three equal masses of m kg each are fixed at the vertices of an equilateral triangle ABC. What is the force on mass m at each vertex?

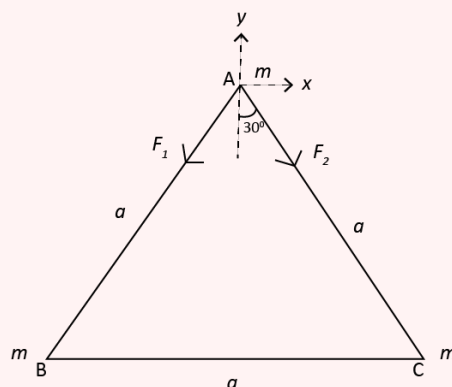


Fig.8.1.7

Solution: Let us consider the forces acting on mass m placed at A.

$$F_1 = F_2 = \frac{Gm^2}{a^2}$$

x components of F_1 and F_2 are equal and opposite hence they cancel out,

y components of F_1 and F_2 are the direction of negative y -axis and get added up

$$F_{\text{net}} = F_1 \cos 30^\circ + F_2 \cos 30^\circ$$

$$= 2 F_1 \cos 30^\circ$$

$$= 2 \frac{Gm^2}{a^2} \sqrt{\frac{3}{2}}$$

$$= \sqrt{3} \frac{Gm^2}{a^2}$$

F_{net} is in the direction of the negative y -axis i.e. along the perpendicular bisector of BC pointing toward BC.

Similarly the forces experienced by the masses at B and C are each equal to $\sqrt{3} \frac{Gm^2}{a^2}$ in magnitude, and they act respectively along the bisectors of angles B and C, into the triangle.

It is interesting to note that these forces will intersect at the centroid of the equilateral triangle ABC, and the mutual angles between each pair will be 120° . So, they will be in equilibrium.

8.1.4 Summary

- **Kepler's laws of planetary motion are**

First law (law of orbits): All planets move in elliptical orbits with the Sun at one of the foci.

Second law (law of areas): The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.

Third Law (law of periods): The square of the time period of revolution of a planet is proportional to the cube of the semi-major axis of the elliptical orbit of the planet.

- **Newton's law of gravitation**

Every object in the universe attracts every other object with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

This force is always attractive.

Development Team

Atul Mody, Former Associate Professor, VES College, Mumbai

V.S. Upadhaya, Former Associate Professor, G.N. Khalsa college, Mumbai

Review Team

C.K. Ghosh, Former Director, NCIDE, IGNOU, New Delhi

Naresh Kumar Sharma, Physics Teacher, Govt. Boys Sr. Secondary School, Bawana, Delhi

Nidhi Kalucha, PGT, Sanskriti School, Chanakyapuri, New Delhi

Vikrant Bhardwaj, PGT, Kendriya Vidyalaya, Gole Market, New Delhi

Member- Coordinator

Shashi Prabha, Professor, CIET, NCERT, New Delhi

Technical support

Riya Rani, JPF, CIET, NCERT, New Delhi

Bhawana Bhatt, Technical Content Developer, CIET, NCERT, New Delhi

Rajat Kumar, Graphic Designer, CIET, NCERT, New Delhi