

Module 2 Gravitational Field

This module is designed to help you to:

- describe motion of objects in gravitational field;
- compare the values of acceleration due to gravity above and below the surface of earth;
- explain the nature of variation of acceleration due to gravity with latitude;
- describe the concept of gravitational potential energy and gravitational potential;
- derive expressions for escape speed on earth, moon and other planets.

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8.2.1 Introduction

This module discusses the use of Newton's law of gravitation to find acceleration due to gravity and gravitational potential energy. The acceleration due to gravity is the intensity of the gravitational field at a point. Gravitational potential energy is due to the position of an object in the gravitational field. If the total energy of an object is negative the object will be bound. If an amount of kinetic energy is supplied to the object by way of imparting a velocity such that the total energy becomes greater than zero, the object is no longer bound and will escape from the influence of the field. The minimum velocity imparted to an object so as to enable it to escape the gravitational field is called escape velocity at that position.

8.2.2 Acceleration due to gravity of the earth.

The magnitude of the force of gravity on a particle of mass m on the surface of the earth is given by

$$F = \frac{GM_E m}{R_E^2} \quad (8.2.1)$$

Where M_E is the mass of the earth and R_E is the radius of the earth.

The acceleration experienced by the mass m near surface of earth is

$$g = \frac{F}{m} = \frac{GM_E}{R_E^2} \quad (8.2.2)$$

Thus the acceleration due to gravity experienced is independent of the mass m .

8.2.3 Objects falling from the same height take the same time to reach the ground

Using kinematical equation

$$s = ut + \frac{1}{2}gt^2$$

$$u = 0 \text{ for both objects}$$

$$h = \frac{1}{2}gt^2$$

Since h and g are the same for both objects irrespective of their mass, t is the same for both objects.

Now let us find the expressions for g above and below the surface of the earth.

8.2.4 Acceleration due to gravity at a height h above the surface of the earth

The distance between the center of the earth and the mass m is $(R_E + h)$. Refer Fig.8.2.1.

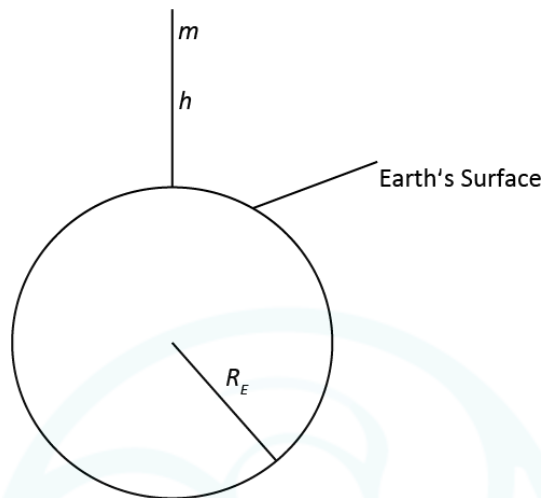


Fig.8.2.1: g at a height h above the surface of the earth

Then the force between the earth and mass m is given by

$$F(h) = \frac{GM_E m}{(R_E + h)^2} \quad (8.2.3)$$

The acceleration experienced by the mass m at a height h above the surface of earth is

$$g_h = \frac{F(h)}{m} = \frac{GM_E}{(R_E + h)^2} \quad (8.2.4)$$

If we consider $h \ll R_E$

$$\text{Then, } g_h = \frac{GM_E}{(R_E + h)^2} = \frac{GM_E}{R_E^2 (1 + h/R_E)^2} = g \left(1 + \frac{h}{R_E}\right)^{-2} \quad (8.2.5)$$

From binomial theorem, $[(1 + x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots]$

If $x \ll 1$, then, higher power terms of x can be neglected and then $(1 + x)^n \approx 1 + nx$

$$\text{Hence } g_h \approx g \left(1 - \frac{2h}{R_E}\right) \quad (8.2.6)$$

Therefore, $g_h < g$.

Thus as we go up above the surface of earth the value of g decreases.

8.2.5 Acceleration due to gravity at a depth d below the surface of the earth

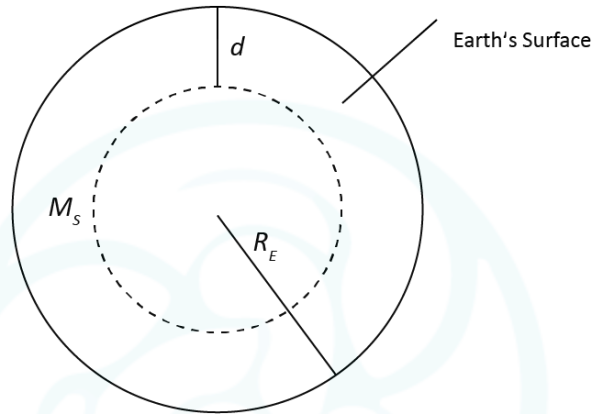


Fig: 8.2.2: g at a depth d below the surface of earth.

In this case the whole earth can be considered to be composed of a smaller sphere of radius $(R_E - d)$ and a spherical shell of thickness d (Fig.8.2.2).

Now the force on m due to the outer shell of thickness d is zero because a body which is inside a spherical shell does not experience a force due to it.

However, the mass m experiences a force due to the spherical part of the earth with radius $(R_E - d)$ as the mass m lies on this surface. Thus, the force is given by

$$F(d) = \frac{GM_s m}{(R_E - d)^2} \quad (8.2.7)$$

where M_s is the mass of the sphere of radius $(R_E - d)$.

Assuming that the earth has uniform density, M_s

We have $M_s = \frac{4}{3}\pi\rho(R_E - d)^3$ where ρ is the density of the earth.

Mass of the earth can be given by

$$M_E = \frac{4}{3} \pi R_E^3 \rho$$

$$\frac{M_s}{M_E} = \left(\frac{R_E - d}{R_E} \right)^3 \quad (8.2.8)$$

Substituting the value of M_s in the Eq(8.2.7), we get

$$F(d) = \frac{GM_E m (R_E - d)}{R_E^3} \quad (8.2.9)$$

Hence the acceleration due to gravity is given by

$$g_d = \frac{GM_E (R_E - d)}{R_E^3} = g \left(\frac{R_E - d}{R_E} \right) = g \left(1 - \frac{d}{R_E} \right) \quad (8.2.10)$$

$$g_d < g$$

Thus as we go down below the earth's surface the acceleration due to gravity decreases by a factor $\left(1 - \frac{d}{R_E} \right)$. At the center of the earth $d = R_E$ and hence $g = 0$

Let us sketch a figure to show variation of g below and above the surface of the earth (Fig.8.2.3).

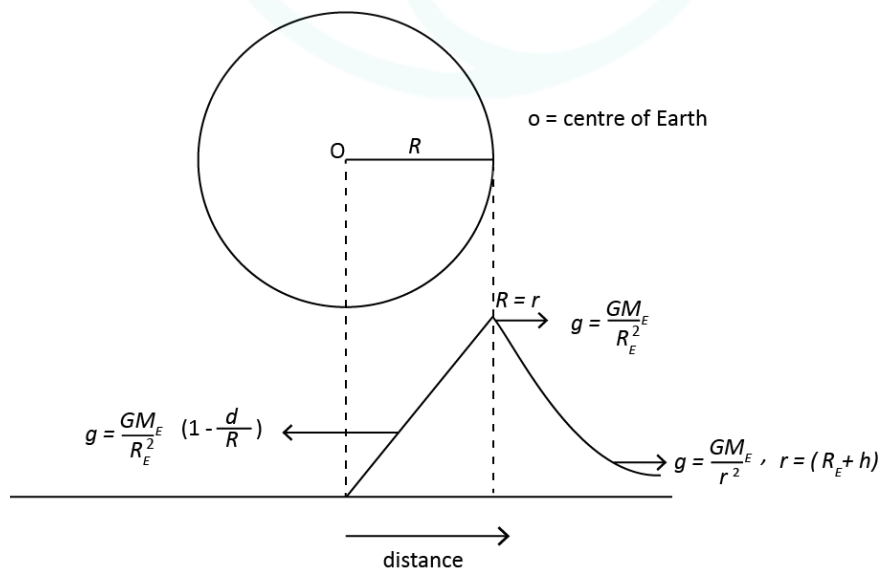


Fig.8.2.3: Variation of g below and above the surface of the earth versus distance from the centre of the earth

Example 8.2.1 A body weighs 63 N on the surface of the earth. What is the gravitational force on it due to the earth at the height equal to half the radius of earth?

Solution: Weight of the body on earth $W = mg = 63 \text{ N}$

Weight of the body at the height h , $W' = mg'$

Since $g' = g/[1 + (h/R_e)]^2$

$$h = R_e/2$$

$$g' = g/[1 + (1/2)]^2 = (4/9)g$$

$$W' = (4/9)W = (4/9) \times 63 = 28 \text{ N}.$$

Example 8.2.2 Assuming the earth to be a sphere of uniform mass density, how much would a body weigh half way down to the center of the earth if it weighs 250 N on the surface?

Solution: Weight of the body on earth $W = mg = 250 \text{ N}$

Weight of the body at the depth d , $W' = mg'$

Since $g' = g[1 - (d/R_e)]$

$$d = R_e/2$$

$$g' = g/[1 - (1/2)] = g/2$$

$$W' = (1/2)W = (1/2) \times 250 = 125 \text{ N}$$

8.2.6 Variation of g with latitude

O is the centre of earth. Its axis of rotation is the line joining the N-pole and S-pole. A mass m is at P. OP is the vertical at P (Fig.8.2.4). Consider a plane made by earth's north-south axes and line OP. In the plane angle made by OP with the equatorial line denoted by λ is the latitude of P. The mass m at P rotates along the circumference of a small circle of radius $R \sin(90^\circ - \lambda) = R \cos \lambda$.

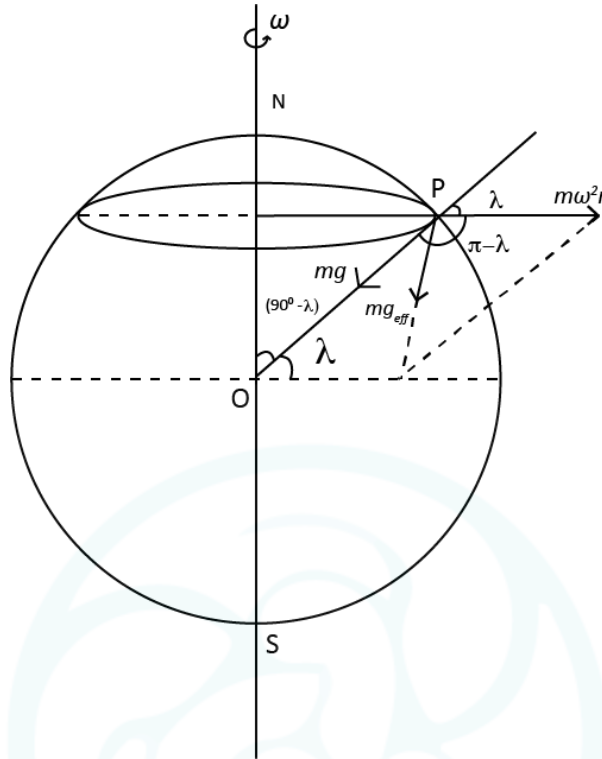


Fig.8.2.4: Variation of g with latitude

So, the centrifugal force experienced by m is $m\omega^2 R \cos \lambda$.

Its component in the direction OP is $m\omega^2 R \cos^2 \lambda$. It acts opposite to the gravitational acceleration.

$\therefore$ The resultant force towards the centre of earth

$$F = mg - m\omega^2 R \cos^2 \lambda$$

$$F = mg \left(1 - \frac{\omega^2 R \cos^2 \lambda}{g} \right) \quad (8.2.11)$$

where g = the acceleration due to gravity on the surface of earth.

$\therefore$ The effective value of g at P is given by

$$g_{eff} = g \left(1 - \frac{\omega^2 R \cos^2 \lambda}{g} \right) \quad (8.2.12)$$

$$g_{eff} < g$$

Gravitational acceleration on the surface of the earth is a function of latitude . Substituting constant quantities and average radius R in the equation,

$$\omega = \frac{2\pi}{24 \times 60 \times 60} \text{ rad s}^{-1}$$

$R = 6400 \text{ km}$ (neglecting the equatorial bulge) $= 6.4 \times 10^6 \text{ m}$ and $g = 9.8 \text{ m/s}^2$, we get

$$\therefore g_{eff} = g \left(1 - \frac{\cos^2 \lambda}{290} \right) \quad (8.2.13)$$

Hence, $\frac{g}{\omega^2 R} = 290$ (apx.)

As we move from the equator towards the pole, g increases, $\cos^2 \lambda$ decreases and hence g_{eff} increases. It is to be noted here that the direction of g_{eff} is not perpendicular to the spherical surface, rather at the time of cooling, earth settled down with its surface perpendicular to the direction of g_{eff} , making its shape oblate. If the surface of the earth is not perpendicular to g_{eff} then there will be an unbalanced component along the surface and the ocean will flow toward the equator.



Let Us Reflect

Q.8.2.1 Consider a planet having a radius half that of earth, but mass is same as that of earth, the acceleration due to gravity of this planet would be

- (A) 19.6 ms^{-2}
- (B) 9.8 ms^{-2}
- (C) 37.2 ms^{-2}
- (D) 4.9 ms^{-2}



Answers

A.8.2.1 (C)

Acceleration due to gravity on earth, $g = \frac{GM}{R^2}$

Acceleration due to gravity on planet

$$g' = \frac{GM}{\left(\frac{R}{2}\right)^2} = 4 \frac{GM}{R^2} = 4g = 4 \times 9.8 \text{ m s}^{-2}$$

$$= 37.2 \text{ m s}^{-2}$$

8.2.7 Gravitational potential energy

Gravitational potential energy of a mass m is the work done to bring the mass to the given position in space from infinity, without acceleration. Zero of potential energy is chosen arbitrarily as per our convenience. Due to the inverse square nature of the gravitational force, we find it convenient to take potential energy at large distances (infinity) to be zero.

Let M be the mass of the earth and m be the mass of the particle at a distance r from the centre of the earth (Fig.8.2.5).

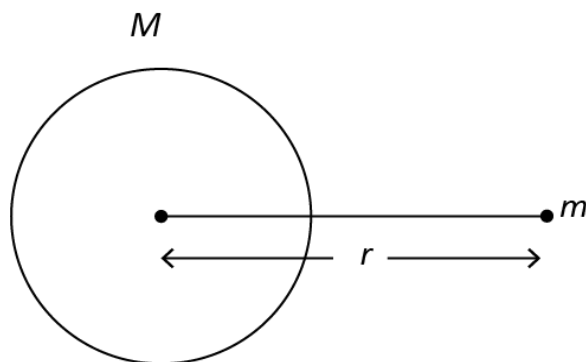


Fig.8.2.5: Gravitational potential energy at a distance r from the centre of the earth

Then the force of gravitation on the particle of mass m directed towards the centre of the earth is given by

$$\vec{F} = - \frac{GMm}{r^2} \hat{r} \quad (8.2.14)$$

Now let us calculate the work done in bringing a particle from $r = r_2$ to $r = r_1$. The work done for an infinitesimal displacement $\vec{dr}$ is $\vec{F} \cdot \vec{dr}$ (Fig.8.2.6). Here $\vec{dr} = -\hat{r}dr$

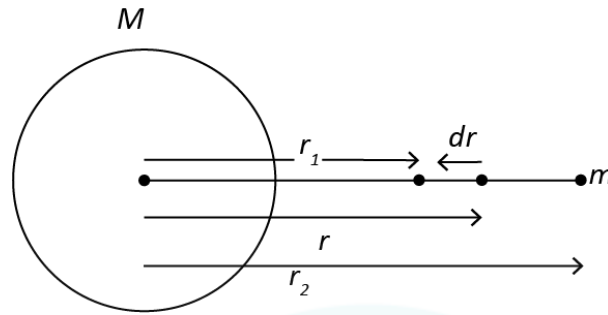


Fig.8.2.6: Work done in bringing a particle from $r = r_2$ to $r = r_1$

$$\therefore dW = -\frac{GMm}{r^2}(\hat{r}) \cdot (-\hat{r}dr) = \frac{GMm}{r^2}dr$$

$$dW = \frac{GMm}{r^2}dr \quad (8.2.15)$$

$$\therefore W = \int_{r_2}^{r_1} \frac{GMm}{r^2}dr = GMm \left(\frac{-1}{r} \right)_{r_2}^{r_1} = -GMm \left(\left(\frac{1}{r_1} \right) - \left(\frac{1}{r_2} \right) \right)$$

When $r_2 = \infty$ and $r_1 = r$

W = potential energy of mass m .

$$\therefore PE = -\frac{GMm}{r} \quad (8.2.16)$$

Potential energy is a scalar quantity. For a system of masses, total potential energy of the system is the sum of potential energy of each pair.

8.2.8 Gravitational potential

Gravitational potential at a point in the gravitational field is defined as the work done per unit mass to bring it from infinity to the given position in space, without acceleration. Gravitational potential is gravitational potential energy per unit mass.

Putting $m = 1$ in the potential energy expression, we get gravitational potential at a distance r from mass M is given by

$$V(r) = \frac{-GM}{r} \quad (8.2.17)$$

The acceleration due to gravity at the given point can be calculated using gravitational potential. Since potential V depends only on the distance r . The relationship between gravitational potential and acceleration due to gravity is $g = \frac{-dV}{dr}$ (8.2.18)

We can easily verify above relation using gravitational potential

$$-\frac{dV}{dr} = -\frac{d}{dr}\left(\frac{-Gm}{r}\right) = -Gm\left(-\frac{1}{r^2}\right)$$

Or, $\frac{Gm}{r^2} = g$

NOTE: Potential and potential energy are similar quantities; potential can be thought of as the potential energy of unit mass (per unit mass). They differ by one dimension in mass. This can be understood from the expressions given by Eqs(8.2.16) and (8.2.17).

Example 8.2.3: A system of four identical particles of mass m each are placed at the vertices of a square of side a (Fig.8.2.7). Find (i) the potential energy of the system and (ii) the potential at the centre of the square.

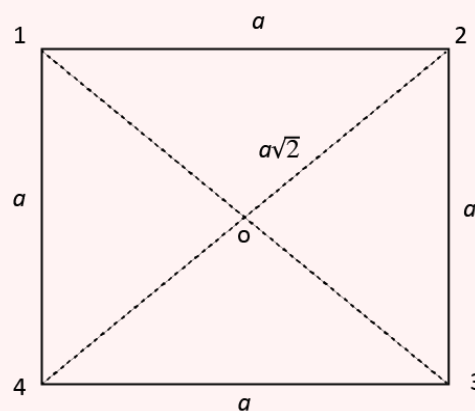


Fig.8.2.7

Solution:

(i) The potential energy of the system is given by

$$U = -G \left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_1 m_4}{r_{14}} + \frac{m_2 m_3}{r_{23}} + \frac{m_2 m_4}{r_{24}} + \frac{m_3 m_4}{r_{34}} \right)$$

$$r_{12} = r_{14} = r_{23} = r_{34} = a$$

$$r_{13} = r_{24} = a\sqrt{2}$$

$$U = -\frac{Gm^2}{a} \left(1 + \frac{1}{\sqrt{2}} + 1 + 1 + \frac{1}{\sqrt{2}} + 1 \right)$$

$$U = -\frac{Gm^2}{a} (4 + \sqrt{2})$$

(ii) The potential at the center of the square is given by

$$V = -G \left(\frac{m_1}{r_{01}} + \frac{m_2}{r_{02}} + \frac{m_3}{r_{03}} + \frac{m_4}{r_{04}} \right)$$

$$r_{01} = r_{02} = r_{03} = r_{04} = \frac{a}{\sqrt{2}}$$

$$V = -\frac{Gm}{a} (\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2})$$

$$V = -\frac{Gm}{a} (4\sqrt{2})$$



Let Us Reflect

Q.8.2.2 Choose the correct alternative:

(a) If the zero of potential energy is at infinity, the total energy of an orbiting satellite is negative of its:

(i) kinetic energy (ii) potential energy.

(b) The energy required to launch an orbiting satellite out of earth's gravitational influence is

(i) more than the energy required to project a stationary object at the same height (as the satellite) out of earth's influence.

(ii) less than the energy required to project a stationary object at the same height (as the satellite) out of earth's influence.

Q.8.2.3 A body of mass 2500 kg is raised to a height $h = \frac{R}{6}$ from the Earth's surface. Find increase in its gravitational potential energy. Given $R = 6400$ km, mass of Earth $= 6 \times 10^{24}$ kg.



Answers

A.8.2.2 (a) (i) (b) (ii).

A.8.2.3 Increase in kinetic energy

$$\begin{aligned}
 &= -\frac{GMm}{R + \frac{R}{6}} - \left(-\frac{GMm}{R}\right) \\
 &= -\frac{6GMm}{7R} + \frac{GMm}{R} \\
 &= \frac{GMm}{7R} \\
 &= 0.22 \times 10^{14}.
 \end{aligned}$$

8.2.9 Escape speed

Escape speed of a body is the minimum speed of projection which allows a body to move far from the earth's surface so that it escapes the gravitational influence of the earth.

Let us consider a body of mass m to which is imparted a speed v . If we throw an object with certain speed, its speed decreases as it goes up. If it becomes zero before the gravitational field of the earth vanishes, it will

come back. However, if a body is projected with escape speed, it will stop at a distance where the gravitational field just vanishes (i.e. at infinity). At infinity, the potential energy of the body is zero. Hence total energy ($E = KE + PE$) of the body would become zero.

Total energy at the surface of the earth,

$$E = \frac{1}{2}mv^2 + \left(-\frac{GMm}{R_E}\right) \quad (8.2.19)$$

In order to escape, E must be non-negative.

$$\frac{1}{2}mv^2 + \left(-\frac{GMm}{R_E}\right) \geq 0$$

The minimum speed to satisfy above condition is (for equality)

$$v_{\text{escape}} = \sqrt{\frac{2GM_E}{R_E}} \quad (8.2.20)$$

$$\therefore g = \frac{GM_E}{R_E^2},$$

$$\therefore v_{\text{escape}} = \sqrt{2gR_E} \quad (8.2.21)$$

Using the value of g and R_E , numerically, $v_{\text{escape}} = 11.2$ km/s. This is the value of escape speed from the surface of the earth.

It is evident that the escape speed does not depend on the mass of the object thrown. However, as g depends on the mass M_E and radius R_E , the escape speed for other planets and the moon will be different from that for the earth.



Let Us Reflect

Q.8.2.4 An object of mass m_1 , which is under the influence of the gravitational field of two planets of masses m_1 and m_2 lies midway between them, the distance between the centre of the planet being d . For what minimum speed of projection at this position, the mass will escape to infinity?



Answers

A.8.2.4 Potential energy of the mass due to the planets

$$= -\frac{Gm_1m}{\frac{d}{2}} = -\frac{Gm_2m}{\frac{d}{2}}$$

$$= -\frac{2Gm}{d}(m_1 + m_2)$$

if the mass is projected with a speed v then its KE = $\frac{1}{2}mv^2$

So, its total mechanical energy

$$E = -\frac{2Gm}{d}(m_1 + m_2) + \frac{1}{2}mv^2$$

in order to escape $E \geq 0$

$$\text{or } \frac{1}{2}mv^2 \geq \frac{2Gm}{d}(m_1 + m_2)$$

$$\therefore v_{\min} = \sqrt{\frac{4G}{d}(m_1 + m_2)}$$

$$\text{or } v_{\min} = 2\sqrt{\frac{G}{d}(m_1 + m_2)}.$$

8.2.10 Summary

- Acceleration due to gravity on the earth's surface is

$$g = \frac{GM_E}{R_E^2}$$

- As we move away from the surface of the earth, the value of g decreases. Acceleration due to gravity at a height h is

$$g_h = \frac{GM_E}{(R_E + h)^2}$$

For $h \ll R_E$

$$g_h = g\left(1 - \frac{2h}{R_E}\right)$$

- As we go down the surface of the Earth the value of g decreases. Acceleration due to gravity at a depth d is

$$g_d = g\left(1 - \frac{d}{R_E}\right)$$

- The value of g is modified due to rotation of the earth on its axis. Acceleration due to gravity at a latitude λ is

$$g_{eff} = g\left(1 - \frac{\omega^2 R \cos^2 \lambda}{g}\right)$$

- Gravitational potential energy of a mass near the surface of the earth is due to the attractive force of the earth. Gravitational potential energy of mass m at distance r is

$$PE = -\frac{GMm}{r}$$

- Gravitational potential at a point is

$$V(r) = \frac{-GM}{r}$$

- For the body to escape from the gravitational influence of the earth, it must be imparted kinetic energy. Escape speed on the surface of the earth

$$v_{\text{escape}} = \sqrt{\frac{2GM_E}{R_E}}$$

Development Team

Atul Mody, Former Associate Professor, VES College, Mumbai

V.S. Upadhaya, Former Associate Professor, G.N. Khalsa college, Mumbai

Review Team

C.K. Ghosh, Former Director, NCIDE, IGNOU, New Delhi

Naresh Kumar Sharma, Physics Teacher, Govt. Boys Sr. Secondary School, Bawana, Delhi

Nidhi Kalucha, PGT, Sanskriti School, Chanakyapuri, New Delhi

Vikrant Bhardwaj, PGT, Kendriya Vidyalaya, Gole Market, New Delhi

Member- Coordinator

Shashi Prabha, Professor, CIET, NCERT, New Delhi

Technical support

Riya Rani, JPF, CIET, NCERT, New Delhi

Bhawana Bhatt, Technical Content Developer, CIET, NCERT, New Delhi

Rajat Kumar, Graphic Designer, CIET, NCERT, New Delhi