

Fraction (mathematics)

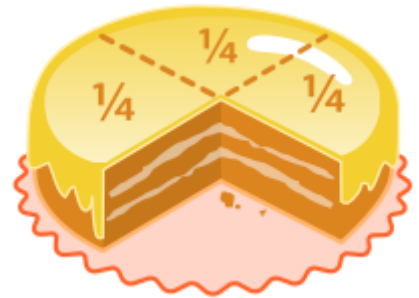
A **fraction** (from Latin *fractus*, "broken") represents a part of a whole or, more generally, any number of equal parts. When spoken in everyday English, a fraction describes how many parts of a certain size there are, for example, one-half, eight-fifths, three-quarters. A *common*, *vulgar*, or *simple* fraction (examples: $\frac{1}{2}$ and $\frac{17}{3}$) consists of an integer numerator displayed above a line (or before a slash), and a non-zero integer denominator; displayed below (or after) that line. Numerators and denominators are also used in fractions that are not *common*, including compound fractions, complex fractions, and mixed numerals.

The numerator represents a number of equal parts, and the denominator, which cannot be zero, indicates how many of those parts make up a unit or a whole. For example, in the fraction $\frac{3}{4}$, the numerator, 3, tells us that the fraction represents 3 equal parts, and the denominator, 4, tells us that 4 parts make up a whole. The picture to the right illustrates $\frac{3}{4}$ or $\frac{3}{4}$ of a cake.

Fractional numbers can also be written without using explicit numerators or denominators, by using decimals, percent signs, or negative exponents (as in 0.01, 1%, and 10^{-2} respectively, all of which are equivalent to $\frac{1}{100}$). An integer such as the number 7 can be thought of as having an implicit denominator of one: 7 equals $\frac{7}{1}$.

Other uses for fractions are to represent ratios and division.^[1] Thus the fraction $\frac{3}{4}$ is also used to represent the ratio 3:4 (the ratio of the part to the whole) and the division $3 \div 4$ (three divided by four).

In mathematics the set of all numbers that can be expressed in the form a/b , where a and b are integers and b is not zero, is called the set of rational numbers and is represented by the symbol **Q**, which stands for quotient. The test for a number being a rational number is that it can be written in that form (i.e., as a common fraction). However, the word *fraction* is also used to describe mathematical expressions that are not rational numbers, for example algebraic fractions (quotients of algebraic expressions), and expressions that contain irrational numbers, such as $\sqrt{2}/2$ (see square root of 2) and $\pi/4$ (see proof that π is irrational).



A cake with one quarter (one fourth) removed. The remaining three fourths are shown. Dotted lines indicate where the cake may be cut in order to divide it into equal parts. Each fourth of the cake is denoted by the fraction $\frac{1}{4}$.

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Vocabulary

In a fraction, the number of equal parts being described is the **numerator** (from Latin *numerātor*, "counter" or "numberer"), and the type or variety of the parts is the **denominator** (from Latin *dēnōminātor*, "thing that names or designates").^[2] As an example, the fraction $\frac{8}{5}$ amounts to eight parts, each of which is of the type named "fifth." In terms of division, the numerator corresponds to the dividend, and the denominator corresponds to the divisor.

Informally, the numerator and denominator may be distinguished by placement alone but in formal contexts they are always separated by a **fraction bar**. The fraction bar may be horizontal (as in $\frac{1}{3}$), oblique (as in $\frac{1}{5}$), or diagonal (as in $\frac{1}{9}$).^[3] These marks are respectively known as the horizontal bar, the slash (US) or stroke (UK), the division slash, and the fraction slash.^[n 1] In typography, horizontal fractions are also known as "en" or "nut fractions" and diagonal fractions as "em fractions", based on the width of a line they take up.^[3]

The denominators of English fractions are generally expressed as ordinal numbers, in the plural if the numerator is not one. (For example, $\frac{2}{5}$ and $\frac{3}{5}$ are both read as a number of "fifths".) Exceptions include the denominator 2, which is always read "half" or "halves", the denominator 4, which may be alternatively expressed as "quarter"/"quarters" or as "fourth"/"fourths", and the denominator 100, which may be alternatively expressed as "hundredth"/"hundredths" or percent. When the denominator is 1, it may be expressed in terms of "wholes" but is more commonly ignored, with the numerator read out as a whole number. (For example, $\frac{3}{1}$ may be described as "three wholes" or as simply "three".) When the numerator is one, it may be omitted. (For example, "a tenth" or "each quarter".)

The entire fraction may be expressed as a single composition, in which case it is hyphenated, or as a number of fractions with a numerator of one, in which case they are not. (For example, "two-fifths" is the fraction $\frac{2}{5}$ and "two fifths" is the same fraction understood as 2 instances of $\frac{1}{5}$.) Fractions should always be hyphenated when used as adjectives. Alternatively, a fraction may be described by reading it out as the numerator "over" the denominator, with the denominator expressed as a cardinal number. (For example, $\frac{3}{1}$ may also be expressed as "three over one".) The term "over" is used even in the case of solidus fractions, where the numbers are placed left and right of a slash mark. (For example, $\frac{1}{2}$ may be read "one-half", "one half", or "one over two".) Fractions

with large denominators that are *not* powers of ten are often rendered in this fashion (e.g., $\frac{1}{117}$ as "one over one hundred seventeen") while those with denominators divisible by ten are typically read in the normal ordinal fashion (e.g., $\frac{6}{1000000}$ as "six-millionths", "six millionths", or "six one-millionths").

Forms of fractions

Simple, common, or vulgar fractions

A **simple fraction** (also known as a **common fraction** or **vulgar fraction**) is a rational number written as a/b or $\frac{a}{b}$, where a and b are both integers.^[7] As with other fractions, the denominator b cannot be zero. Examples include $\frac{1}{2}$, $-\frac{8}{5}$, $\frac{-8}{5}$, $\frac{8}{-5}$, and $3/17$. *Simple fractions* can be positive or negative, proper, or improper (see below). Compound fractions, complex fractions, mixed numerals, and decimals (see below) are not *simple fractions*, though, unless irrational, they can be evaluated to a simple fraction.

- A unit fraction is a common fraction with a numerator of 1, e.g. $\frac{1}{7}$. Unit fractions can also be expressed using negative exponents, as in 2^{-1} , which represents $1/2$, and 2^{-2} , which represents $1/(2^2)$ or $1/4$.
- A dyadic fraction is a common fraction in which the denominator is a power of two, e.g. $\frac{1}{8} = \frac{1}{2^3}$.

Proper and improper fractions

Common fractions can be classified as either proper or improper. When the numerator and the denominator are both positive, the fraction is called proper if the numerator is less than the denominator, and improper otherwise.^{[8][9]} In general, a common fraction is said to be a **proper fraction** if the absolute value of the fraction is strictly less than one—that is, if the fraction is greater than -1 and less than 1 .^{[10][11]} It is said to be an **improper fraction**, or sometimes **top-heavy fraction**^[12] if the absolute value of the fraction is greater than or equal to 1. Examples of proper fractions are $2/3$, $-3/4$, and $4/9$; examples of improper fractions are $9/4$, $-4/3$, and $3/3$.

Reciprocals and the "invisible denominator"

The **reciprocal** of a fraction is another fraction with the numerator and denominator exchanged. The reciprocal of $\frac{3}{7}$, for instance, is $\frac{7}{3}$. The product of a fraction and its reciprocal is 1, hence the reciprocal is the multiplicative inverse of a fraction. The reciprocal of a proper fraction is improper, and the reciprocal of an improper fraction not equal to 1, that is, numerator and denominator are not equal, is a proper fraction.

When the numerator and denominator of a fraction are equal ($\frac{7}{7}$, for example), its value is 1, and the fraction therefore is improper. Its reciprocal also has the value 1, and is improper too.

Any integer can be written as a fraction with the number one as denominator. For example, 17 can be written as $\frac{17}{1}$, where 1 is sometimes referred to as the *invisible denominator*. Therefore, every fraction or integer, except for zero, has a reciprocal. The reciprocal of 17 is $\frac{1}{17}$.

Ratios

A ratio is a relationship between two or more numbers that can be sometimes expressed as a fraction. Typically, a number of items are grouped and compared in a ratio, specifying numerically the relationship between each group. Ratios are expressed as "group 1 to group 2 ... to group n ". For example, if a car lot had 12 vehicles, of which

- 2 are white,
- 6 are red, and
- 4 are yellow

then the ratio of red to white to yellow cars is 6 to 2 to 4. The ratio of yellow cars to white cars is 4 to 2 and may be expressed as 4:2 or 2:1.

A ratio is often converted to a fraction when it is expressed as a ratio to the whole. In the above example, the ratio of yellow cars to all the cars on the lot is 4:12 or 1:3. We can convert these ratios to a fraction and say that $\frac{4}{12}$ of the cars or $\frac{1}{3}$ of the cars in the lot are yellow. Therefore, if a person randomly chose one car on the lot, then there is a one in three chance or probability that it would be yellow.

Decimal fractions and percentages

A **decimal fraction** is a fraction whose denominator is not given explicitly, but is understood to be an integer power of ten. Decimal fractions are commonly expressed using decimal notation in which the implied denominator is determined by the number of digits to the right of a decimal separator, the appearance of which (e.g., a period, a raised period (•), a comma) depends on the locale (for examples, see decimal separator). Thus for 0.75 the numerator is 75 and the implied denominator is 10 to the second power, viz. 100, because there are two digits to the right of the decimal separator. In decimal numbers greater than 1 (such as 3.75), the fractional part of the number is expressed by the digits to the right of the decimal (with a value of 0.75 in this case). 3.75 can be written either as an improper fraction, $\frac{375}{100}$, or as a mixed number $3\frac{75}{100}$.

Decimal fractions can also be expressed using scientific notation with negative exponents, such as 6.023×10^{-7} , which represents 0.0000006023. The 10^{-7} represents a denominator of 10^7 . Dividing by 10^7 moves the decimal point 7 places to the left.

Decimal fractions with infinitely many digits to the right of the decimal separator represent an infinite series. For example, $\frac{1}{3} = 0.333\dots$ represents the infinite series $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots$.

Another kind of fraction is the percentage (Latin *per centum* meaning "per hundred", represented by the symbol %), in which the implied denominator is always 100. Thus, 51% means $\frac{51}{100}$. Percentages greater than 100 or less than zero are treated in the same way, e.g. 311% equals $\frac{311}{100}$, and -27% equals $-\frac{27}{100}$.

The related concept of *permille* or *parts per thousand* (ppt) has an implied denominator of 1000, while the more general parts-per notation, as in *parts per million* (ppm), means that the proportion is $\frac{75}{1,000,000}$.

Whether common fractions or decimal fractions are used is often a matter of taste and context. Common fractions are used most often when the denominator is relatively small. By mental calculation, it is easier to multiply 16 by $\frac{3}{16}$ than to do the same calculation using the fraction's decimal equivalent (0.1875). And it is more accurate to multiply 15 by $\frac{1}{3}$, for example, than it is to multiply 15 by any decimal approximation of one third. Monetary values are commonly expressed as decimal fractions with denominator 100, i.e., with two decimals, for example \$3.75. However, as noted above, in pre-decimal British currency, shillings and pence were often given the form (but not the meaning) of a fraction, as, for example $\frac{3}{6}$ (read "three and six") meaning 3 shillings and 6 pence, and having no relationship to the fraction $\frac{3}{6}$.

Mixed numbers

A **mixed numeral** (also called a *mixed fraction* or *mixed number*) is a traditional denotation of the sum of a non-zero integer and a proper fraction (having the same sign). It is used primarily in measurement $2\frac{3}{16}$ inches, for example. Scientific measurements almost invariably use decimal notation rather than mixed numbers. The sum is implied without the use of a visible operator such as the appropriate "+". For example, in referring to two entire cakes and three quarters of another cake, the numerals denoting the integer part and the fractional part of the cakes are written next to each other as $2\frac{3}{4}$ instead of the unambiguous notation $2 + \frac{3}{4}$. Negative mixed numerals, as in $-2\frac{3}{4}$, are treated like $-(2 + \frac{3}{4})$. Any such sum of a *whole* plus a *part* can be converted to an improper fraction by applying the rules of adding unlike quantities.

This tradition is, formally, in conflict with the notation in algebra where adjacent factors denote a product, without an explicit infix operator. When two algebraic expressions are written next to each other, the operation of multiplication is implied by this general rule: $2x$ always means the product of 2 and x , even if the value of x is a fraction. The expression $2\frac{b}{c}$ for example is not a mixed

number, instead, multiplication is expressly required, where $2\frac{b}{c} = 2 \cdot \frac{b}{c}$.

For better readability the multiplication is sometimes made explicit or parentheses are added. So $2\frac{b}{c}$ may be written as

$$2 \cdot \frac{b}{c}, \quad \text{or} \quad 2 \times \frac{b}{c}, \quad \text{or} \quad 2 \left(\frac{b}{c} \right), \dots$$

An improper fraction can be converted to a mixed number as follows:

1. Divide the numerator by the denominator. In the example, $\frac{11}{4}$, divide 11 by 4. $11 \div 4 = 2$ with remainder 3.
2. The quotient (without the remainder) becomes the whole number part of the mixed number. The remainder becomes the numerator of the fractional part. In the example, 2 is the whole number part and 3 is the numerator of the fractional part.
3. The new denominator is the same as the denominator of the improper fraction. In the example, they are both 4. Thus: $\frac{11}{4} = 2\frac{3}{4}$.

Historical notions

Egyptian fraction

An Egyptian fraction is the sum of distinct positive unit fractions, for example $\frac{1}{2} + \frac{1}{3}$. This definition derives from the fact that the ancient Egyptians expressed all fractions except $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$ in this manner. Every positive rational number can be expanded as an Egyptian fraction. For example, $\frac{5}{7}$ can be written as $\frac{1}{2} + \frac{1}{6} + \frac{1}{21}$. Any positive rational number can be written as a sum of unit fractions in infinitely many ways. Two ways to write $\frac{13}{17}$ are $\frac{1}{2} + \frac{1}{4} + \frac{1}{68}$ and $\frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{68}$.

'Complex' and 'Compound' fractions

Both notions are outdated^[13] and nowadays used in no well defined manner, partly even taken synonymously for each other^[14] or for mixed numerals.^[15] They lost their meaning as technical terms and the attributes "complex" and "compound" tend to be used in their every day meaning of "consisting of parts".

- Complex fractions

Not to be confused with fractions involving complex numbers

In a **complex fraction**, either the numerator, or the denominator, or both, is a fraction or a mixed number,^{[16][17]} corresponding to division of fractions. For example, $\frac{\frac{1}{2}}{\frac{1}{3}}$ and $\frac{12\frac{3}{4}}{26}$ are complex fractions. To reduce a complex fraction to a simple fraction, treat the

longest fraction line as representing division. For example:

$$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2} = 1\frac{1}{2}$$

$$\frac{12\frac{3}{4}}{26} = 12\frac{3}{4} \cdot \frac{1}{26} = \frac{12 \cdot 4 + 3}{4} \cdot \frac{1}{26} = \frac{51}{4} \cdot \frac{1}{26} = \frac{51}{104}$$

$$\frac{\frac{3}{2}}{5} = \frac{3}{2} \times \frac{1}{5} = \frac{3}{10}$$

$$\frac{8}{\frac{1}{3}} = 8 \times \frac{3}{1} = 24.$$

If, in a complex fraction, there is no unique way to tell which fraction line takes precedence, then this expression is improperly formed, because of ambiguity. So $5/10/20/40$ is not a valid mathematical expression, because of multiple possible interpretations, e.g. as

$$5/(10/(20/40)) = \frac{5}{10/\frac{20}{40}} = \frac{1}{4} \quad \text{or as} \quad (5/10)/(20/40) = \frac{\frac{5}{10}}{\frac{20}{40}} = 1$$

▪ Compound fractions

A **compound fraction** is a fraction of a fraction, or any number of fractions connected with the word *of*,^{[16][17]} corresponding to multiplication of fractions. To reduce a compound fraction to a simple fraction, just carry out the multiplication (see the section on multiplication). For example, $\frac{3}{4}$ of $\frac{5}{7}$ is a compound fraction, corresponding to $\frac{3}{4} \times \frac{5}{7} = \frac{15}{28}$. The terms compound fraction and complex fraction are closely related and sometimes one is used as a synonym for the other. (For example, the compound fraction $\frac{3}{4} \times \frac{5}{7}$ is equivalent to the complex fraction $\frac{3/4}{7/5}$.)

Arithmetic with fractions

Like whole numbers, fractions obey the commutative, associative, and distributive laws, and the rule against division by zero

Equivalent fractions

Multiplying the numerator and denominator of a fraction by the same (non-zero) number results in a fraction that is equivalent to the original fraction. This is true because for any non-zero number n , the fraction $\frac{n}{n} = 1$. Therefore, multiplying by $\frac{n}{n}$ is equivalent to multiplying by one, and any number multiplied by one has the same value as the original number. By way of an example, start with the fraction $\frac{1}{2}$. When the numerator and denominator are both multiplied by 2, the result is $\frac{2}{4}$, which has the same value (0.5) as $\frac{1}{2}$. To picture this visually imagine cutting a cake into four pieces; two of the pieces together ($\frac{2}{4}$) make up half the cake ($\frac{1}{2}$).

Dividing the numerator and denominator of a fraction by the same non-zero number will also yield an equivalent fraction. This is called reducing or simplifying the fraction. A simple fraction in which the numerator and denominator are coprime (that is, the only positive integer that goes into both the numerator and denominator evenly is 1) is said to be irreducible, in lowest terms, or in simplest terms. For example, $\frac{3}{9}$ is not in lowest terms because both 3 and 9 can be exactly divided by 3. In contrast, $\frac{3}{8}$ is in lowest terms—the only positive integer that goes into both 3 and 8 evenly is 1.

Using these rules, we can show that $\frac{5}{10} = \frac{1}{2} = \frac{10}{20} = \frac{50}{100}$.

A common fraction can be reduced to lowest terms by dividing both the numerator and denominator by their greatest common divisor. For example, as the greatest common divisor of 63 and 462 is 21, the fraction $\frac{63}{462}$ can be reduced to lowest terms by dividing the numerator and denominator by 21:

$$\frac{63}{462} = \frac{63 \div 21}{462 \div 21} = \frac{3}{22}$$

The Euclidean algorithm gives a method for finding the greatest common divisor of any two positive integers.

Comparing fractions

Comparing fractions with the same positive denominator yields the same result as comparing the numerators:

$$\frac{3}{4} > \frac{2}{4} \text{ because } 3 > 2, \text{ and the equal denominators } 4 \text{ are positive.}$$

If the equal denominators are negative, then the opposite result of comparing the numerators holds for the fractions:

$$\frac{3}{-4} < \frac{2}{-4} \text{ because } \frac{a}{-b} = \frac{-a}{b} \text{ and } -3 < -2.$$

If two positive fractions have the same numerator, then the fraction with the smaller denominator is the larger number. When a whole is divided into equal pieces, if fewer equal pieces are needed to make up the whole, then each piece must be larger. When two positive fractions have the same numerator, they represent the same number of parts, but in the fraction with the smaller denominator, the parts are larger.

One way to compare fractions with different numerators and denominators is to find a common denominator. To compare $\frac{a}{b}$ and $\frac{c}{d}$, these are converted to $\frac{a \cdot d}{b \cdot d}$ and $\frac{b \cdot c}{b \cdot d}$. Then bd is a common denominator and the numerators ad and bc can be compared. This modification of the two fractions is known as "cross multiplying", and it is not necessary to determine the value of the common denominator to compare fractions – one can just compare ad and bc , without evaluating bd , e.g., comparing $\frac{2}{3} ? \frac{1}{2}$ gives $\frac{4}{6} > \frac{3}{6}$.

For the more laborious question $\frac{5}{18} ? \frac{4}{17}$, multiply top and bottom of each fraction by the denominator of the other fraction, to get a common denominator, yielding $\frac{5 \times 17}{18 \times 17} ? \frac{18 \times 4}{18 \times 17}$. It is not necessary to calculate 18×17 – only the numerators need to be compared. Since $5 \times 17 (= 85)$ is greater than $4 \times 18 (= 72)$, the result of comparing is $\frac{5}{18} > \frac{4}{17}$.

Because every negative number, including negative fractions, is less than zero, and every positive number, including positive fractions, is greater than zero, it follows that any negative fraction is less than any positive fraction. This allows, together with the above rules, to compare all possible fractions.

Addition

The first rule of addition is that only like quantities can be added; for example, various quantities of quarters. Unlike quantities, such as adding thirds to quarters, must first be converted to like quantities as described below: Imagine a pocket containing two quarters, and another pocket containing three quarters; in total, there are five quarters. Since four quarters is equivalent to one (dollar), this can be represented as follows:

$$\frac{2}{4} + \frac{3}{4} = \frac{5}{4} = 1\frac{1}{4}.$$

Adding unlike quantities

To add fractions containing unlike quantities (e.g. quarters and thirds), it is necessary to convert all amounts to like quantities. It is easy to work out the chosen type of fraction to convert to; simply multiply together the two denominators (bottom number) of each fraction. In case of an integer number apply the invisible denominator 1.

For adding quarters to thirds, both types of fraction are converted to twelfths, thus:

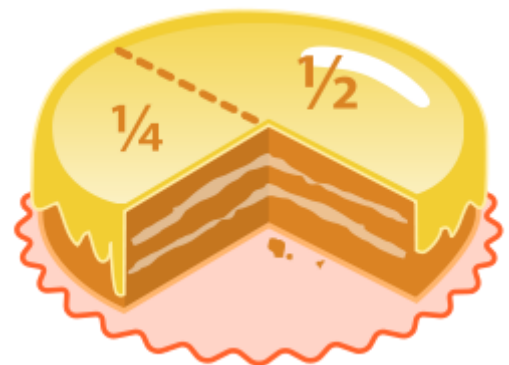
$$\frac{1}{4} + \frac{1}{3} = \frac{1 \times 3}{4 \times 3} + \frac{1 \times 4}{3 \times 4} = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}.$$

Consider adding the following two quantities:

$$\frac{3}{5} + \frac{2}{3}$$

First, convert $\frac{3}{5}$ into fifteenths by multiplying both the numerator and denominator by three: $\frac{3}{5} \times \frac{3}{3} = \frac{9}{15}$. Since $\frac{3}{3}$ equals 1, multiplication by $\frac{3}{3}$ does not change the value of the fraction.

Second, convert $\frac{2}{3}$ into fifteenths by multiplying both the numerator and denominator by five: $\frac{2}{3} \times \frac{5}{5} = \frac{10}{15}$.



If $\frac{1}{2}$ of a cake is to be added to $\frac{1}{4}$ of a cake, the pieces need to be converted into comparable quantities, such as cake-eighths or cake-quarters.

Now it can be seen that:

$$\frac{3}{5} + \frac{2}{3}$$

is equivalent to:

$$\frac{9}{15} + \frac{10}{15} = \frac{19}{15} = 1\frac{4}{15}$$

This method can be expressed algebraically:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

And for expressions consisting of the addition of three fractions:

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{a(df) + c(bf) + e(bd)}{bdf}$$

This method always works, but sometimes there is a smaller denominator that can be used (a least common denominator). For example, to add $\frac{3}{4}$ and $\frac{5}{12}$ the denominator 48 can be used (the product of 4 and 12), but the smaller denominator 12 may also be used, being the least common multiple of 4 and 12.

$$\frac{3}{4} + \frac{5}{12} = \frac{9}{12} + \frac{5}{12} = \frac{14}{12} = \frac{7}{6} = 1\frac{1}{6}$$

Subtraction

The process for subtracting fractions is, in essence, the same as that of adding them: find a common denominator, and change each fraction to an equivalent fraction with the chosen common denominator. The resulting fraction will have that denominator, and its numerator will be the result of subtracting the numerators of the original fractions. For instance,

$$\frac{2}{3} - \frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$$

Multiplication

Multiplying a fraction by another fraction

To multiply fractions, multiply the numerators and multiply the denominators. Thus:

$$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$$

To explain the process, consider one third of one quarter. Using the example of a cake, if three small slices of equal size make up a quarter, and four quarters make up a whole, twelve of these small, equal slices make up a whole. Therefore, a third of a quarter is a twelfth. Now consider the numerators. The first fraction, two thirds, is twice as large as one third. Since one third of a quarter is one twelfth, two thirds of a quarter is two twelfths. The second fraction, three quarters, is three times as large as one quarter, so two thirds of three quarters is three times as large as two thirds of one quarter. Thus two thirds times three quarters is six twelfths.

A short cut for multiplying fractions is called "cancellation". Effectively the answer is reduced to lowest terms during multiplication. For example:

$$\frac{2}{3} \times \frac{3}{4} = \frac{\cancel{2}^1}{\cancel{3}_1} \times \frac{\cancel{3}^1}{\cancel{4}_2} = \frac{1}{1} \times \frac{1}{2} = \frac{1}{2}$$

A two is a common factor in both the numerator of the left fraction and the denominator of the right and is divided out of both. Three is a common factor of the left denominator and right numerator and is divided out of both.

Multiplying a fraction by a whole number

Since a whole number can be rewritten as itself divided by 1, normal fraction multiplication rules can still apply

$6 \times \frac{3}{4} = \frac{6}{1} \times \frac{3}{4} = \frac{18}{4}$ This method works because the fraction 6/1 means six equal parts, each one of which is a whole.

Multiplying mixed numbers

When multiplying mixed numbers, it is considered preferable to convert the mixed number into an improper fraction. For example:

$$3 \times 2\frac{3}{4} = 3 \times \left(\frac{8}{4} + \frac{3}{4}\right) = 3 \times \frac{11}{4} = \frac{33}{4} = 8\frac{1}{4}$$

In other words, $2\frac{3}{4}$ is the same as $\frac{8}{4} + \frac{3}{4}$, making 11 quarters in total (because 2 cakes, each split into quarters makes 8 quarters total) and 33 quarters is $8\frac{1}{4}$, since 8 cakes, each made of quarters, is 32 quarters in total.

Division

To divide a fraction by a whole number, you may either divide the numerator by the number, if it goes evenly into the numerator, or multiply the denominator by the number. For example, $\frac{10}{3} \div 5$ equals $\frac{2}{3}$ and also equals $\frac{10}{3 \cdot 5} = \frac{10}{15}$, which reduces to $\frac{2}{3}$. To divide a number by a fraction, multiply that number by the reciprocal of that fraction. Thus $\frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \times \frac{4}{3} = \frac{1 \cdot 4}{2 \cdot 3} = \frac{2}{3}$.

Converting between decimals and fractions

To change a common fraction to a decimal, do a long division of the decimal representations of the numerator by the denominator (this is idiomatically also phrased as "divide the denominator into the numerator"), and round the answer to the desired accuracy. For example, to change $\frac{1}{4}$ to a decimal, divide **1.00** by **4** ("**4** into **1.00**"), to obtain **0.25**. To change $\frac{1}{3}$ to a decimal, divide **1.000...** by **3** ("**3** into **1.0000...**"), and stop when the desired accuracy is obtained, e.g., at **4** decimals with **0.3333**. Note that $\frac{1}{4}$ can be written exactly with two decimal digits, while the fraction $\frac{1}{3}$ cannot be written exactly as a decimal with a finite number of digits. To change a decimal to a fraction, write in the denominator a **1** followed by as many zeroes as there are digits to the right of the decimal point, and write in the numerator all the digits of the original decimal, just omitting the decimal point. Thus $12.3456 = \frac{123456}{10000}$.

Converting repeating decimals to fractions

Decimal numbers, while arguably more useful to work with when performing calculations, sometimes lack the precision that common fractions have. Sometimes an infinite repeating decimal is required to reach the same precision. Thus, it is often useful to convert repeating decimals into fractions.

The preferred way to indicate a repeating decimal is to place a bar over the digits that repeat, for example $\overline{0.789} = 0.789789789...$ For repeating patterns where the repeating pattern begins immediately after the decimal point, a simple division of the pattern by the same number of nines as numbers it has will suffice. For example:

$$\begin{aligned} 0.\overline{5} &= 5/9 \\ 0.\overline{62} &= 62/99 \\ 0.\overline{264} &= 264/999 \end{aligned}$$

$$0.\overline{6291} = 6291/9999$$

In case leading zeros precede the pattern, the nines are suffixed by the same number of trailing zeros:

$$0.0\overline{5} = 5/90$$

$$0.000\overline{392} = 392/999000$$

$$0.001\overline{2} = 12/9900$$

In case a non-repeating set of decimals precede the pattern (such as $0.1523\overline{987}$), we can write it as the sum of the non-repeating and repeating parts, respectively:

$$0.1523 + 0.0000\overline{987}$$

Then, convert both parts to fractions, and add them using the methods described above:

$$1523/10000 + 987/9990000 = 1522464/9990000$$

Alternatively, algebra can be used, such as below:

1. Let x = the repeating decimal:

$$x = 0.1523\overline{987}$$

2. Multiply both sides by the power of 10 just great enough (in this case 10^4) to move the decimal point just before the repeating part of the decimal number:

$$10,000x = 1,523\overline{987}$$

3. Multiply both sides by the power of 10 (in this case 10^7) that is the same as the number of places that repeat:

$$10,000,000x = 1,523,987\overline{987}$$

4. Subtract the two equations from each other (if $a = b$ and $c = d$, then $a - c = b - d$):

$$10,000,000x - 10,000x = 1,523,987\overline{987} - 1,523.\overline{987}$$

5. Continue the subtraction operation to clear the repeating decimal:

$$9,990,000x = 1,523,987 - 1,523$$

$$9,990,000x = 1,522,464$$

6. Divide both sides by 9,990,00 to represent x as a fraction

$$x = 1522464/9990000$$

Fractions in abstract mathematics

In addition to being of great practical importance, fractions are also studied by mathematicians, who check that the rules for fractions given above are consistent and reliable. Mathematicians define a fraction as an ordered pair (a, b) of integers a and $b \neq 0$, for which the operations addition, subtraction, multiplication, and division are defined as follows:^[18]

$$(a, b) + (c, d) = (ad + bc, bd)$$

$$(a, b) - (c, d) = (ad - bc, bd)$$

$$(a, b) \cdot (c, d) = (ac, bd)$$

$$(a, b) \div (c, d) = (ad, bc) \quad (\text{with, additionally, } c \neq 0)$$

These definitions agree in every case with the definitions given above; only the notation is different. Alternatively, instead of defining subtraction and division as operations, the "inverse" fractions with respect to addition and multiplication might be defined as:

$$\begin{aligned} -(a, b) &= (-a, b) && \text{additive inverse fractions,} \\ &&& \text{with } (0, b) \text{ as additive unities, and} \\ (a, b)^{-1} &= (b, a) && \text{multiplicative inverse fractions, for } a \neq 0, \\ &&& \text{with } (b, b) \text{ as multiplicative unities.} \end{aligned}$$

Furthermore, the relation, specified as

$$(a, b) \sim (c, d) \iff ad = bc,$$

is an equivalence relation of fractions. Each fraction from one equivalence class may be considered as a representative for the whole class, and each whole class may be considered as one abstract fraction. This equivalence is preserved by the above defined operations, i.e., the results of operating on fractions are independent of the selection of representatives from their equivalence class. Formally, for addition of fractions

$$\begin{aligned} (a, b) \sim (a', b') \quad \text{and} \quad (c, d) \sim (c', d') \quad \text{imply} \\ ((a, b) + (c, d)) \sim ((a', b') + (c', d')) \end{aligned}$$

and similarly for the other operations.

In case of fractions of integers the fractions with (a, b) coprime are often taken as uniquely determined representatives for their *equivalent* fractions, which are considered to be the *same* rational number. This way the fractions of integers make up the field of the rational numbers.

More generally, a and b may be elements of any integral domain R , in which case a fraction is an element of the field of fractions of R . For example, polynomials in one indeterminate, with coefficients from some integral domain D , are themselves an integral domain, call it P . So for a and b elements of P , the generated *field of fractions* is the field of rational fractions (also known as the field of rational functions).

Algebraic fractions

An algebraic fraction is the indicated quotient of two algebraic expressions. As with fractions of integers, the denominator of an algebraic fraction cannot be zero. Two examples of algebraic fractions are $\frac{3x}{x^2 + 2x - 3}$ and $\frac{\sqrt{x+2}}{x^2 - 3}$. Algebraic fractions are subject to the same field properties as arithmetic fractions.

If the numerator and the denominator are polynomials, as in $\frac{3x}{x^2 + 2x - 3}$, the algebraic fraction is called a **rational fraction** (or **rational expression**). An **irrational fraction** is one that is not rational, as, for example, one that contains the variable under a fractional exponent or root, as in $\frac{\sqrt{x+2}}{x^2 - 3}$.

The terminology used to describe algebraic fractions is similar to that used for ordinary fractions. For example, an algebraic fraction is in lowest terms if the only factors common to the numerator and the denominator are 1 and -1. An algebraic fraction whose numerator or denominator, or both, contain a fraction, such as $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$, is called a **complex fraction**.

The field of rational numbers is the field of fractions of the integers, while the integers themselves are not a field but rather an integral domain. Similarly, the rational expressions are the field of fractions of polynomials. There are different integral domains of polynomials, depending on the integral domain the coefficients of the polynomials are from (e.g. from integers, real numbers,

complex numbers ...). Considering the field of fractions generated by polynomials with real coefficients, radical expressions such as $\sqrt{2}/2$ are also rational fractions, as is the transcendental expression $\pi/2$, since all of $\sqrt{2}$, π , and 2 are (constant) polynomials over the *reals*. These same expressions, however, would not be considered elements of the field of fractions generated by polynomials with *integer* coefficients. This specific field would contain just the 2 of the three polynomials above, or $1/2$ as fraction, but no radical or transcendental expressions.

The term partial fraction is used when decomposing rational expressions into sums. The goal is to write the rational expression as the sum of other rational expressions with denominators of lesser degree. For example, the rational expression $\frac{2x}{x^2 - 1}$ can be rewritten as the sum of two fractions: $\frac{1}{x+1} + \frac{1}{x-1}$. This is useful in many areas such as integral calculus and differential equations.

Radical expressions

A fraction may also contain radicals in the numerator and/or the denominator. If the denominator contains radicals, it can be helpful to rationalize it (compare Simplified form of a radical expression), especially if further operations, such as adding or comparing that fraction to another, are to be carried out. It is also more convenient if division is to be done manually. When the denominator is a monomial square root, it can be rationalized by multiplying both the top and the bottom of the fraction by the denominator:

$$\frac{3}{\sqrt{7}} = \frac{3}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$$

The process of rationalization of binomial denominators involves multiplying the top and the bottom of a fraction by the conjugate of the denominator so that the denominator becomes a rational number. For example:

$$\begin{aligned} \frac{3}{3-2\sqrt{5}} &= \frac{3}{3-2\sqrt{5}} \cdot \frac{3+2\sqrt{5}}{3+2\sqrt{5}} = \frac{3(3+2\sqrt{5})}{3^2 - (2\sqrt{5})^2} = \frac{3(3+2\sqrt{5})}{9-20} = -\frac{9+6\sqrt{5}}{11} \\ \frac{3}{3+2\sqrt{5}} &= \frac{3}{3+2\sqrt{5}} \cdot \frac{3-2\sqrt{5}}{3-2\sqrt{5}} = \frac{3(3-2\sqrt{5})}{3^2 - (2\sqrt{5})^2} = \frac{3(3-2\sqrt{5})}{9-20} = -\frac{9-6\sqrt{5}}{11} \end{aligned}$$

Even if this process results in the numerator being irrational, like in the examples above, the process may still facilitate subsequent manipulations by reducing the number of irrationals one has to work with in the denominator.

Typographical variations

In computer displays and typography, simple fractions are sometimes printed as a single character, e.g. $\frac{1}{2}$ (one half). See the article on Number Forms for information on doing this in Unicode.

Scientific publishing distinguishes four ways to set fractions, together with guidelines on use.^[19]

- **special fractions:** fractions that are presented as a single character with a slanted bar with roughly the same height and width as other characters in the text. Generally used for simple fractions, such as: $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{4}$, and $\frac{3}{4}$. Since the numerals are smaller, legibility can be an issue, especially for small-sized fonts. These are not used in modern mathematical notation, but in other contexts.
- **case fractions:** similar to special fractions, these are rendered as a single typographical character but with a horizontal bar, thus making them *upright*. An example would be $\frac{1}{2}$, but rendered with the same height as other characters. Some sources include all rendering of fractions as *case fractions* if they take only one typographical space, regardless of the direction of the bar.^[20]
- **shilling or solidus fractions:** $1/2$, so called because this notation was used for pre-decimal British currency (£sd), as in $2/6$ for a half crown, meaning two shillings and six pence. While the notation "two shillings and six pence" did not represent a fraction, the forward slash is now used in fractions, especially for fractions inline with prose (rather than displayed), to avoid uneven lines. It is also used for fractions within fractions (complex fractions) or within exponents to increase legibility. Fractions written this way, also known as *piece fractions*,^[21] are written all on one typographical line, but take 3 or more typographical spaces.

- **built-up fractions:** $\frac{1}{2}$. This notation uses two or more lines of ordinary text, and results in a variation in spacing between lines when included within other text. While large and legible, these can be disruptive, particularly for simple fractions or within complex fractions.

History

The earliest fractions were reciprocals of integers: ancient symbols representing one part of two, one part of three, one part of four, and so on.^[22] The Egyptians used Egyptian fractions c. 1000 BC. About 4000 years ago, Egyptians divided with fractions using slightly different methods. They used least common multiples with unit fractions. Their methods gave the same answer as modern methods.^[23] The Egyptians also had a different notation for dyadic fractions in the Akhmim Wooden Tablet and several Rhind Mathematical Papyrus problems.

The Greeks used unit fractions and (later) continued fractions. Followers of the Greek philosopher Pythagoras (c. 530 BC) discovered that the square root of two cannot be expressed as a fraction of integers. (This is commonly though probably erroneously ascribed to Hippasus of Metapontum, who is said to have been executed for revealing this fact.) In 150 BC Jain mathematicians in India wrote the "Sthananga Sutra", which contains work on the theory of numbers, arithmetical operations, and operations with fractions.

A modern expression of fractions known as **bhinnarasi** seems to have originated in India in the work of Aryabhatta (c. AD 500), Brahmagupta (c. 628), and Bhaskara (c. 1150).^[24] Their works form fractions by placing the numerators (Sanskrit: *amsa*) over the denominators (*cheda*), but without a bar between them.^[24] In Sanskrit literature, fractions were always expressed as an addition to or subtraction from an integer. The integer was written on one line and the fraction in its two parts on the next line. If the fraction was marked by a small circle (◦) or cross (+), it is subtracted from the integer; if no such sign appears, it is understood to be added. For example, Bhaskara I writes^[25]

६	१	२
१	१	१◦
४	५	९

which is the equivalent of

6	1	2
1	1	-1
4	5	9

and would be written in modern notation as $\frac{1}{4}$, $1\frac{1}{5}$, and $2-\frac{1}{9}$ (i.e., $1\frac{8}{9}$).

The horizontal fraction bar is first attested in the work of Al-Hassār (fl. 1200),^[24] a Muslim mathematician from Fez, Morocco, who specialized in Islamic inheritance jurisprudence. In his discussion he writes, "... for example, if you are told to write three-fifths and a third of a fifth, write thus, $\frac{3}{5} \frac{1}{3}$."^[26] The same fractional notation—with the fraction given before the integer^[24]—appears soon after in the work of Leonardo Fibonacci in the 13th century.^[27]

In discussing the origins of decimal fractions, Dirk Jan Struik states:^[28]

"The introduction of decimal fractions as a common computational practice can be dated back to the Flemish pamphlet *De Thiende*, published at Leyden in 1585, together with a French translation, *La Disme*, by the Flemish mathematician Simon Stevin (1548–1620), then settled in the Northern Netherlands. It is true that decimal fractions were used by the Chinese many centuries before Stevin and that the Persian astronomer Al-Kāshī used both decimal and sexagesimal fractions with great ease in his *Key to arithmetic* (Samarkand, early fifteenth century).^[29]

While the Persian mathematician Jamshīd al-Kāshī claimed to have discovered decimal fractions himself in the 15th century, J. Lennart Berggren notes that he was mistaken, as decimal fractions were first used five centuries before him by the Baghdadi mathematician Abu'l-Hasan al-Uqlidisi as early as the 10th century.^{[30][n 2]}

In formal education

Pedagogical tools

In primary schools, fractions have been demonstrated through Cuisenaire rods, Fraction Bars, fraction strips, fraction circles, paper (for folding or cutting), pattern blocks, pie-shaped pieces, plastic rectangles, grid paper, dot paper, geoboards, counters and computer software.

Documents for teachers

Several states in the United States have adopted learning trajectories from the Common Core State Standards Initiative's guidelines for mathematics education. Aside from sequencing the learning of fractions and operations with fractions, the document provides the following definition of a fraction: "A number expressible in the form $\frac{a}{b}$ where *a* is a whole number and *b* is a positive whole number. (The word *fraction* in the standards always refers to a non-negative number)."^[32] The document itself also refers to negative fractions.

See also

- Continued fraction
- 0.999...
- Multiple

Notes

- Some typographers such as Bringinghurst mistakenly distinguish the slash(/) as the virgule and the fraction slash(/) as the solidus,^[4] although in fact both are synonyms for the standard slash.^{[5][6]}
- While there is some disagreement among history of mathematics scholars as to the primacy of al-Uqlidisi's contribution, there is no question as to his major contribution to the concept of decimal fractions.^[31]

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