

2.4 Multiplication of Integers

Recall that multiplication is defined as repeated addition from elementary school. For example, $5 \cdot 6 = 6 \cdot 5 = 30$, since:

$$5 \cdot 6 = 6 + 6 + 6 + 6 + 6 = 30$$

$$6 \cdot 5 = 5 + 5 + 5 + 5 + 5 = 30$$

To develop a rule for **multiplication of integers**, we must determine what happens when one or both of the numbers is negative. We start with the product $4 \cdot (-6)$. Using the idea of repeated addition:

$$4 \cdot (-6) = (-6) + (-6) + (-6) + (-6) = -24$$

For the product $-6 \cdot 4$, we apply the commutative property of multiplication:

$$-6 \cdot 4 = 4 \cdot (-6) = (-6) + (-6) + (-6) + (-6) = -24$$

When multiplying two numbers of different signs (a positive number and a negative number), it appears that the product is negative. What about the product $(-6) \cdot (-4)$? The repeated addition idea will not work, since we cannot add -4 to itself -6 times, nor can we add -6 to itself -4 times. To determine the answer, consider the sequence of multiplications:

$$(-6) \cdot 3 = -18$$

$$(-6) \cdot 2 = -12$$

$$(-6) \cdot 1 = -6$$

$$(-6) \cdot 0 = 0$$

The pattern on the left should be clear; the numbers being multiplied by -6 are decreasing by 1. On the right the numbers are increasing by 6. Continuing this pattern:

$$\begin{aligned}(-6) \cdot 3 &= -18 \\(-6) \cdot 2 &= -12 \\(-6) \cdot 1 &= -6 \\(-6) \cdot 0 &= 0 \\(-6) \cdot (-1) &= 6 \\(-6) \cdot (-2) &= 12 \\(-6) \cdot (-3) &= 18 \\(-6) \cdot (-4) &= 24\end{aligned}$$

Thus, it appears that multiplying two negative numbers results in a positive number. In summary, when multiplying two numbers, the sign is positive if the two numbers have the same sign, and negative if the two numbers have different signs. This is an easy rule to remember, and thus multiplication does not require the number line to determine the sign of the answer, as does addition.

Example 1 Multiply the two numbers.

- a. $7(-6)$
- b. $-9(8)$
- c. $(-15)(-5)$
- d. $(13)(3)$

Solution

- a. Apply the rule for signs:

$$\begin{aligned}7(-6) &= -(7 \cdot 6) && \text{rule for signs} \\ &= -42 && \text{multiply numbers}\end{aligned}$$
- b. Apply the rule for signs:

$$\begin{aligned}-9(8) &= -(9 \cdot 8) && \text{rule for signs} \\ &= -72 && \text{multiply numbers}\end{aligned}$$

- c. Apply the rule for signs:
 $(-15)(-5) = +(15 \cdot 5)$ rule for signs
 $= 75$ multiply numbers
- d. Apply the rule for signs:
 $(13)(13) = +(13 \cdot 3)$ rule for signs
 $= 39$ multiply numbers

The same properties for multiplication are true, as illustrated in the property box:

Property Name	Property	Example
Commutative property	$a \cdot b = b \cdot a$	$4 \cdot (-8) = -8 \cdot 4$
Associative property	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$	$-2 \cdot (5 \cdot (-7)) = (-2 \cdot 5) \cdot (-7)$
Identity property	$a \cdot 1 = 1 \cdot a = a$	$-15 \cdot 1 = 1 \cdot (-15) = -15$
Multiplication property of 0	$a \cdot 0 = 0 \cdot a = 0$	$-7 \cdot 0 = 0 \cdot (-7) = 0$
Zero factor property	If $a \cdot b = 0$, $a = 0$ or $b = 0$	If $-4 \cdot x = 0$, $x = 0$

When multiplying more than two numbers, we can use the rule of signs to pair up negative numbers, effectively “canceling” the negatives. For example, to compute $(-4)(-1)(2)(-3)(-2)$, a quick survey of the numbers indicates there are four negative numbers being multiplied. Since each pair produces a positive product, the answer must be positive. Thus:

$$(-4)(-1)(2)(-3)(-2) = +(4 \cdot 1 \cdot 2 \cdot 3 \cdot 2) \quad \text{rule for signs}$$

$$= 48 \quad \text{multiply numbers}$$

Example 2 Multiply the following numbers.

- a. $(-5)(-3)(-2)$
- b. $(-4)(-3)(-2)(-1)$
- c. $(-6)(2)(-2)(-1)$
- d. $(-9)(-8)(-13)(0)$

Solution

- a. Pairing pairs of negative numbers indicates the product is negative:

$$\begin{aligned}(-5)(-3)(-2) &= -(5 \cdot 3 \cdot 2) && \text{rule for signs} \\ &= -30 && \text{multiply numbers}\end{aligned}$$

- b. Pairing pairs of negative numbers indicates the product is positive:

$$\begin{aligned}(-4)(-3)(-2)(-1) &= +(4 \cdot 3 \cdot 2 \cdot 1) && \text{rule for signs} \\ &= 24 && \text{multiply numbers}\end{aligned}$$

- c. Pairing pairs of negative numbers indicates the product is negative:

$$\begin{aligned}(-6)(2)(-2)(-1) &= -(6 \cdot 2 \cdot 2 \cdot 1) && \text{rule for signs} \\ &= -24 && \text{multiply numbers}\end{aligned}$$

- d. Notice that one of the numbers being multiplied is 0. By the multiplication property of 0, we know this results in a 0 product. So

$$(-9)(-8)(-13)(0) = 0.$$

Recall that solutions to equations are numbers which, when substituted for the variable in the equation, produce a true statement. We can now determine solutions to equations which involve multiplication, as the following example illustrates. Note that we often write $3x$ to represent the product $3 \cdot x$.

Example 3 Determine whether or not the given integer value is a solution to the equation.

- a. $3x = 9$; $x = 3$
- b. $5x = -25$; $x = -5$
- c. $-4y = -24$; $y = -6$
- d. $-3a = 36$; $a = -12$

Solution

- a. Substitute $x = 3$ into the equation and determine whether the statement is true:

$$3 \cdot 3 = 9$$

$$9 = 9$$

Since this last statement ($9 = 9$) is true, $x = 3$ is a solution to the equation $3x = 9$.

- b. Substitute $x = -5$ into the equation and determine whether the statement is true:

$$5 \cdot (-5) = -25$$

$$-25 = -25$$

Since this last statement ($-25 = -25$) is true, $x = -5$ is a solution to the equation $5x = -25$.

- c. Substitute $y = -6$ into the equation and determine whether the statement is true:

$$-4 \cdot (-6) = -24$$

$$24 = -24$$

Since this last statement ($24 = -24$) is false, $y = -6$ is not a solution to the equation $-4y = -24$.

- d. Substitute $a = -12$ into the equation and determine whether the statement is true:

$$-3 \cdot (-12) = 36$$

$$36 = 36$$

Since this last statement ($36 = 36$) is true, $a = -12$ is a solution to the equation $-3a = 36$.

Previously we studied sequences of numbers which are arithmetic sequences. Recall that these are sequences of numbers in which each term results in adding a fixed number onto the previous term. A second type of sequence is called a **geometric sequence**. These are sequences of numbers in which each term results in *multiplying* the same number (called the **common ratio**) by the previous term. For example, the sequence 2, 4, 8, 16, ... is a geometric sequence, since:

$$2 \cdot 2 = 4$$

$$4 \cdot 2 = 8$$

$$8 \cdot 2 = 16$$

The next term of this sequence is $16 \cdot 2 = 32$.

Example 4 Find the next term in each geometric sequence.

- a. 1, 3, 9, 27, ...
- b. 2, -6, 18, -54, ...
- c. -3, -6, -12, -24, ...
- d. -4, 8, -16, 32, ...

Solution

- a. Note that $1 \cdot 3 = 3$, $3 \cdot 3 = 9$, and $9 \cdot 3 = 27$, so the common ratio is 3. The next term is $27 \cdot 3 = 81$.
- b. Note that $2 \cdot (-3) = -6$, $-6 \cdot (-3) = 18$, and $18 \cdot (-3) = -54$, so the common ratio is -3. The next term is $-54 \cdot (-3) = 162$.
- c. Note that $-3 \cdot 2 = -6$, $-6 \cdot 2 = -12$, and $-12 \cdot 2 = -24$, so the common ratio is 2. The next term is $-24 \cdot 2 = -48$.
- d. Note that $-4 \cdot (-2) = 8$, $8 \cdot (-2) = -16$, and $-16 \cdot (-2) = 32$, so the common ratio is -2. The next term is $32 \cdot (-2) = -64$.

Now that we have completed both addition and multiplication, there is one new property of numbers which involves these two operations. Suppose you are given the expression $5(3 + 4)$ to compute. From the last chapter, we know to add within the parentheses first, then perform the multiplication. Thus:

$$\begin{aligned} 5(3 + 4) &= 5 \cdot 7 \\ &= 35 \end{aligned}$$

Notice, however, a different approach:

$$\begin{aligned} 5(3 + 4) &= 5 \cdot 3 + 5 \cdot 4 \\ &= 15 + 20 \\ &= 35 \end{aligned}$$

This second approach utilizes a new property of numbers, called the **distributive property**:

Distributive Property (addition form): $a(b + c) = a \cdot b + a \cdot c$

Distributive Property (subtraction form): $a(b - c) = a \cdot b - a \cdot c$

This property will be used extensively in algebra.

Example 5 Compute each expression two ways: directly by computing parentheses first, and by using the distributive property.

- a. $6(5 + 9)$
- b. $-4(3 - 7)$
- c. $-6(-2 + 6)$
- d. $-5(-3 - 4)$

Solution

- a. Computing the expression directly:

$$6(5 + 9) = 6 \cdot 14 = 84$$

Computing the expression using the distributive property:

$$6(5 + 9) = 6 \cdot 5 + 6 \cdot 9 = 30 + 54 = 84$$

Note that the two values are the same.

- b. Computing the expression directly:

$$-4(3 - 7) = -4[3 + (-7)] = -4(-4) = 16$$

Computing the expression using the distributive property:

$$-4(3 - 7) = -4 \cdot 3 - (-4) \cdot 7 = -12 - (-28) = -12 + 28 = 16$$

Note that the two values are the same.

- c. Computing the expression directly:

$$-6(-2 + 6) = -6(4) = -24$$

Computing the expression using the distributive property:

$$-6(-2 + 6) = -6 \cdot (-2) + (-6) \cdot 6 = 12 + (-36) = -24$$

Note that the two values are the same.

- d. Computing the expression directly:

$$-5(-3 - 4) = -5[-3 + (-4)] = -5(-7) = 35$$

Computing the expression using the distributive property:

$$-5(-3 - 4) = -5(-3) - (-5)(4) = 15 - (-20) = 15 + 20 = 35$$

Note that the two values are the same.

The distributive property provides us with another rationale of why the product of two negative numbers is a positive number. Consider the product $-5 \cdot (-3)$. Since $4 - 7 = -3$, we can write this product as

$$\begin{aligned} -5 \cdot (-3) &= -5 \cdot (4 - 7) \\ &= -5 \cdot 4 - (-5) \cdot 7 \\ &= -20 - (-35) \\ &= -20 + 35 \\ &= 15 \end{aligned}$$

This is an alternative argument to that of establishing patterns which we did at the beginning of this section.

Terminology

multiplication of integers	commutative property
associative property	identity property
multiplication property of 0	zero factor property
geometric sequence	common ratio
distributive property (addition and subtraction form)	

Exercise Set 2.4

Multiply the two integers.

- | | |
|-----------------------|-----------------------|
| 1. $-4 \cdot 6$ | 2. $-8 \cdot 7$ |
| 3. $6 \cdot (-5)$ | 4. $12 \cdot (-7)$ |
| 5. $-5 \cdot (-6)$ | 6. $-9 \cdot (-12)$ |
| 7. $(-13) \cdot (-8)$ | 8. $(-11) \cdot (-6)$ |
| 9. $0 \cdot (-45)$ | 10. $-53 \cdot (0)$ |

Give the property name which justifies each statement.

- | | |
|---|---|
| 11. $-42 \cdot 8 = 8 \cdot (-42)$ | 12. $-13 \cdot 7 = 7 \cdot (-13)$ |
| 13. $-16 \cdot 0 = 0 \cdot (-16) = 0$ | 14. $-6 \cdot (-7 \cdot (-2)) = (-6 \cdot (-7)) \cdot (-2)$ |
| 15. $-8 \cdot (5 \cdot (-3)) = (-8 \cdot 5) \cdot (-3)$ | 16. $-24 \cdot 0 = 0 \cdot (-24) = 0$ |
| 17. $45(-8 + 15) = 45 \cdot (-8) + 45 \cdot 15$ | 18. $13(9 - 40) = 13 \cdot 9 - 13 \cdot 40$ |

19. $-68 \cdot 1 = 1 \cdot (-68) = -68$ 20. If $7x = 0$, then $x = 0$.
 21. If $-9y = 0$, then $y = 0$. 22. $-14 \cdot 1 = 1 \cdot (-14) = -14$
 23. $-17(-8 - 13) = -17 \cdot (-8) - (-17) \cdot 13$
 24. $-23(-6 + (-8)) = -23 \cdot (-6) + (-23) \cdot (-8)$

Multiply the following integers.

25. $(-5)(3)(-4)$ 26. $(-6)(2)(-3)$
 27. $(-6)(-3)(-2)$ 28. $(-5)(-3)(-4)$
 29. $(8)(-2)(5)$ 30. $(7)(3)(-4)$
 31. $(-2)(-3)(4)(-5)$ 32. $(-6)(-2)(5)(-3)$
 33. $(-4)(-5)(-6)(-1)$ 34. $(-2)(-8)(-5)(-4)$

Determine whether or not the given integer value is a solution to the equation.

35. $4x = -16$; $x = -4$ 36. $4x = -16$; $x = 4$
 37. $-5y = -25$; $y = -5$ 38. $-5y = -25$; $y = 5$
 39. $-6a = 48$; $a = -8$ 40. $-6a = 48$; $a = 8$
 41. $-8x = 0$; $x = 8$ 42. $-8x = 0$; $x = 0$
 43. $-8 + x = 0$; $x = 8$ 44. $-8 + x = 0$; $x = 0$

Find the next term in each geometric sequence.

45. 3, 6, 12, 24, ... 46. 1, 2, 4, 8, ...
 47. 2, -4, 8, -16, ... 48. 2, -6, 18, -54, ...
 49. -5, 10, -20, 40, ... 50. -1, 3, -9, 27, ...
 51. -4, -8, -16, -32, ... 52. -4, -12, -36, -108, ...
 53. -1, 10, -100, 1000, ... 54. 1, -5, 25, -125, ...
 55. 1, -4, 16, -64, ... 56. -1, 1, -1, 1, ...

Compute each expression two ways: directly by computing parentheses first, and by using the distributive property.

57. $3(4 + 8)$ 58. $5(7 + 12)$
 59. $8(7 - 2)$ 60. $6(12 - 5)$
 61. $5(7 - 13)$ 62. $7(5 - 12)$
 63. $-6(4 + 8)$ 64. $-9(7 + 15)$
 65. $-8(8 - 14)$ 66. $-9(5 - 13)$

67. $-12(-6 + 5)$

68. $-8(-9 + 3)$

69. $-4(-8 - 5)$

70. $-7(-6 - 12)$

71. $-15(-16 - 20)$

72. $-25(-12 - 24)$

Answer each question as true or false, where x and y represent integers. If it is false, give a specific example to show that it is false. If it is true, explain why.

73. If $xy < 0$, then $x < 0$ or $y < 0$.

74. If $xy < 0$, then $x < 0$ and $y < 0$.

75. If $xy > 0$, then $x > 0$ or $y > 0$.

76. If $xy > 0$, then $x > 0$ and $y > 0$.

77. If $xy = 0$, then $x = 0$ or $y = 0$.

78. If $xy = 0$, then $x = 0$ and $y = 0$.

79. If $x = 0$, then $xy = 0$.

80. If $x = 0$, then $x + y = 0$.

Answer each question.

81. If the product of -12 and -4 is added to the product of -8 and 5 , what is the result?

82. If the product of -8 and 12 is added to the product of -5 and -12 , what is the result?

83. A new company has monthly losses of \$485 for the first two years. How much is their total loss in the first two years?

84. A new business has weekly losses of \$258 for the first year. How much is their total loss in the first year?

85. If the sum of -12 and -6 is multiplied by the sum of -8 and -4 , what is the result?

86. If the sum of -18 and 7 is multiplied by the sum of -7 and -6 , what is the result?