

The Integers ($\mathbb{Z}$):

- An extension of $\mathbb{N}$ that introduces the negative numbers.
- The nonnegative numbers in $\mathbb{Z}$ are “essentially the same thing as” $\mathbb{N}$ in terms of addition and multiplication.
- Subtraction is now closed. The difference of two integers is always another integer.
- Exact division is not always defined (example: $2 \div 3$).
- The Well-Ordering Principle is no longer true. However...
- *Every* integer now has a unique predecessor and successor. This allows for a modified form of Proof by Induction.

Addition in $\mathbb{Z}$, Algebraic Properties:

- Addition is still closed, associative, and commutative. Zero is still the identity element.
- Each integer now has an *additive inverse*. The additive inverse of a is the unique integer b such that $a + b = 0$. We write this inverse as $-a$.
- Subtraction is best defined in terms of additive inverses: $a - b = a + (-b)$. One consequence is that subtraction is automatically closed.

If we ignore the multiplication operation in $\mathbb{Z}$ and consider abstract properties of addition, we get an example of an *abelian group*.

Multiplication in $\mathbb{Z}$, Algebraic Properties:

- Multiplication is closed, associative, and commutative.
- There is a multiplicative identity (the number 1).
- Most integers do NOT have a multiplicative inverse.
- The Cancellation Law still holds (but we can't cancel zero).
- The *Distributive Law* provides a link between addition and multiplication.

If we consider the abstract properties of addition and multiplication in $\mathbb{Z}$, we get an *integral domain*. A more general algebraic structure is a *ring*.

We will construct $\mathbb{Z}$ in terms of $\mathbb{N}$. One way to do this...

- For each nonzero $x \in \mathbb{N}$, introduce a new object $-x$.
- Formally define the algebraic operations on this extended set. For example, we could define $(-x) + x = 0$ for all $x \in \mathbb{N}$.
- Show that these extended operations satisfy the properties we expect.

There are some potential problems with this approach:

- Do the new objects that we introduce really exist?
- Can we be sure they behave exactly as expected?

Here is a philosophically “better” way (that turns out to be more efficient):

- Assume we are familiar with the basic objects/properties of $\mathbb{N}$, including algebraic operations.
- Define objects in $\mathbb{Z}$ in terms of objects in $\mathbb{N}$, using set-theoretic operations.
- Define the algebraic operations in this new set $\mathbb{Z}$, using the operations from $\mathbb{N}$.

Our main tools will be ordered pairs (which we’ll take for granted) and relations (which are well-defined as sets of ordered pairs).

Building the set $\mathbb{Z}$

- From the abstract point of view, an integer is simply the *difference* between two natural numbers. However, $a, b \in \mathbb{N}$ does not always mean $a - b$ can be defined in $\mathbb{N}$.
- We can get around this by using the ordered pair (a, b) to represent the difference $a - b$. If this sounds strange, keep in mind that we do essentially the same thing with rational numbers $\frac{a}{b}$, even though we don't write them as ordered pairs.
- One problem is that a particular integer (difference) can be represented in various ways. For example, $(0, 1)$, $(1, 2)$, and $(2014, 2015)$ all represent the same integer (which one?)
- To solve this problem, we use equivalence relations/classes.

Equivalence Relations

A binary relation R on a (nonempty) set S is said to be an *equivalence relation* on S if all of the following are true:

- For each $x \in S$, $(x, x) \in R$. [Reflexive]
- If $(x, y) \in R$, then $(y, x) \in R$. [Symmetric]
- If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. [Transitive]

We'll use the xRy notation, instead of $(x, y) \in R$. One reason is that S will be a set of ordered pairs, and nesting ordered pairs gets confusing.

We often write $\sim$ instead of R when dealing with an abstract equivalence relation. See the next slide...

Equivalence Relations, Better Notation

A binary relation $\sim$ on a (nonempty) set S is said to be an *equivalence relation* on S if all of the following are true:

- For each $x \in S$, $x \sim x$. [Reflexive]
- If $x \sim y$, then $y \sim x$. [Symmetric]
- If $x \sim y$ and $y \sim z$, then $x \sim z$. [Transitive]

If the relation $\sim$ is the ordinary $=$ relation, the above properties are satisfied. Congruence (either geometric or “modulo n ”) are also equivalence relations.

An equivalence relation on $\mathbb{N} \times \mathbb{N}$

Returning to the problem of building $\mathbb{Z}$ from $\mathbb{N}$, we want the integer $a - b$ to be represented by the ordered pair (a, b) .

- When does an ordered pair (c, d) represent the same integer?
- Obviously, when $a - b = c - d$. The trick is to do this without mentioning subtraction.
- If we want our usual arithmetic and algebraic properties to work, the above condition can be restated as $a + d = b + c$.
- If a, b, c and d are in $\mathbb{N}$, we'll define $(a, b) \sim (c, d)$ to mean that $a + d = b + c$ [the outer sum equals the inner sum]. As an exercise, verify that this is an equivalence relation (on $\mathbb{N} \times \mathbb{N}$). Do not use subtraction!

Equivalence Classes

Informally speaking, an equivalence relation allows us to take various sets of objects, and regard all objects in a given set as *equivalent*, even if they are not “truly” equal.

- Example: The fractions $\frac{4}{12}$ and $\frac{7}{21}$ appear to be different, but they “represent the same number” when reduced to lowest terms.
- Example: The ordered pairs $(2, 5)$ and $(11, 14)$ are obviously unequal, but they have the same difference (in terms of our earlier discussion, they should represent the same integer).

For any equivalence relation, we can formalize this idea using *equivalence classes*. We’ll explore some abstract properties before returning to the problem of building $\mathbb{Z}$ from $\mathbb{N}$.

Equivalence Classes

Definition. Let $\sim$ be an equivalence relation on the set S . For each $x \in S$, we define the equivalence class of x , written as $[x]$, to be

$$[x] = \{y \in S : y \sim x\}$$

Since any equivalence relation is Reflexive, every equivalence class is nonempty, and every $x \in S$ belongs to at least one equivalence class.

Additionally, we have the following IMPORTANT results:

- **Corollary D-1:** $x \sim y$ if and only if $[x] = [y]$.
- **Corollary D-2:** $[x] \neq [y]$ if and only if $[x] \cap [y] = \emptyset$.

Building the set $\mathbb{Z}$

We are now ready to build $\mathbb{Z}$ from $\mathbb{N}$. As discussed earlier, we define the relation $\sim$ on the set $\mathbb{N} \times \mathbb{N}$ (ordered pairs) using:

$$(a, b) \sim (c, d) \text{ if and only if } a + d = b + c$$

- Since this is an equivalence relation, we have a set of equivalence classes. We define the set $\mathbb{Z}$ to be the set of equivalence classes. *An integer is an equivalence class.*
- To visualize this, draw (in Quadrant I only) the set of lines $y = x - c$, where c is an integer. All points on the line $y = x - c$ are in the same equivalence class; this equivalence class is the integer c .
- We'll discuss arithmetic later. For now, we're just defining a set.

Negative Integers

Note that some equivalence classes contain an ordered pair of the form $(a, 0)$, where $a \in \mathbb{N}$. If we ignore the second coordinate, the set of such equivalence classes is “essentially the same thing as” the set of natural numbers; note that $a - 0 = a$

However, some equivalence classes do not contain a pair $(a, 0)$. These instead contain a pair of the form $(0, a)$. These classes represent will represent the negative integers; note that $0 - a = -a$.

A surprising challenging exercise: Show that any ordered pair (x, y) is equivalent to either a pair of the form $(a, 0)$ or a pair of the form $(0, a)$. Do not use subtraction (this is what makes the exercise difficult).

Addition in $\mathbb{Z}$

Having constructed the set $\mathbb{Z}$, we want define an addition operation. To accomplish this, we'll refer to the addition operation in $\mathbb{N}$.

- If we think of integers as differences, $(a - b) + (c - d) = (a + c) - (b + d)$.
- As ordered pairs, this becomes $(a, b) + (c, d) = (a + c, b + d)$.
- However, an integer isn't really an ordered pair, but an *equivalence class*.
- To see why this might be an issue, try adding the *integers* $2 + 5$ and $(-4) + 3$.

Addition in $\mathbb{Z}$

- 2 and 5 are integers, but also natural numbers. It had better be true that $2 + 5 = 7$.
- Indeed, $(2, 0)$ represents the integer 2, and $(5, 0)$ represents the integer 5. Adding these ordered pairs (as defined previously), we get $(2 + 5, 0 + 0) = (7, 0)$. This represents the integer 7, so everything seems fine.
- However, $(5, 3)$ is another representation of the integer 2, and $(10, 5)$ is another representation of the integer 5. If we add these together, we do not get $(7, 0)$.
- Although the result is $(15, 8)$, this does represent the same integer, 7. We need to ensure that this will always be the case.

Exercises

Prove each of the following, WITHOUT using negatives/subtraction. Addition of ordered pairs is defined coordinate-wise. Any variables that appear are assumed to be natural numbers.

- If $(a, b) \sim (x, y)$ and $(c, d) \sim (w, z)$, then $(a, b) + (c, d) \sim (x, y) + (w, z)$.
- $(a, b) + (z, z) \sim (a, b)$.
- $(a, b) + (b, a) \sim (z, z)$.

Question: The second and third items are describing what familiar properties of addition?

Negation of ordered pairs

Although there is no negation operation within $\mathbb{N}$

...an integer represents a difference: (a, b) is a representation of $a - b$. The negation of a difference uses the same two values, but reverses the order.

We can thus define negation and subtraction of ordered pairs:

- Negation: $-(a, b) = (b, a)$.
- Subtraction: $(a, b) - (c, d) = (a, b) + [-(c, d)] = (a, b) + (d, c)$.

Abstract subtraction is commonly defined this way, as addition of a negative.

Exercises: Negation/Subtraction in $\mathbb{Z}$

Prove the following without using negation/subtraction within $\mathbb{N}$:

- $(a, b) \sim (c, d)$ if and only if $-(a, b) \sim -(c, d)$.
- $-(- (a, b)) \sim (a, b)$
- $(a, b) \sim (b, a)$ if and only if $(a, b) \sim (z, z)$.
- $-((a, b) - (c, d)) \sim (c, d) - (a, b)$.

Operations on equivalence classes

We often have an operation on some set S , and want to define a similar relation on the set of equivalence classes (using a particular equivalence relation on S). The general process goes like this:

1. For each equivalence class, choose a “representative” of that class (some object contained in the class).
2. Perform the operation (in S) on the representative(s) chosen in the previous step. This gives an object in S .
3. Take the equivalence class of the resulting object.

For this to be successful, we need to know that our final result (equivalence class) does not depend on the choice of representative(s) in Step 1.

Further Exercises (time permitting)

- In order to be consistent with ordinary multiplication in $\mathbb{N}$, how would you define multiplication of ordered pairs? This is not as easy as it might seem! For example, $(0, 1) \times (0, 1) \neq (0, 1)$. Why not?
- How can you define the product $(a, b) \times (c, d)$, in order that multiplication works as expected (hint: each ordered pair represents a difference).
- In terms of ordered pairs, what does it mean to say that the square of any integer is a never negative? Prove that this is true.
- In terms of ordered pairs, what does it mean to say that $(-1) \times m = -m$? Prove that this is true.