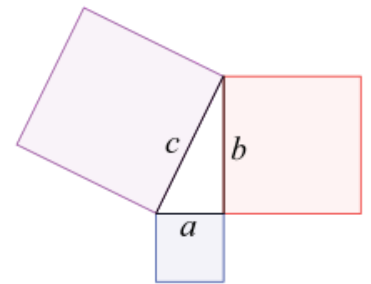


Pythagorean triple

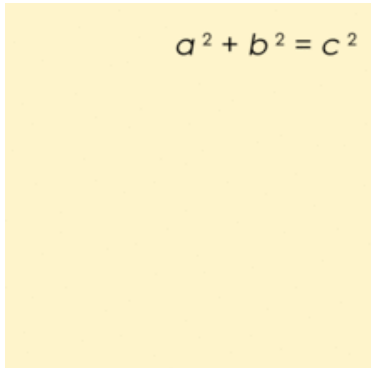
A **Pythagorean triple** consists of three positive integers *a*, *b*, and *c*, such that $a^2 + b^2 = c^2$. Such a triple is commonly written (*a*, *b*, *c*), and a well-known example is (3, 4, 5). If (*a*, *b*, *c*) is a Pythagorean triple, then so is (*ka*, *kb*, *kc*) for any positive integer *k*. A **primitive Pythagorean triple** is one in which *a*, *b* and *c* are coprime (that is, they have no common divisor larger than 1).^[1] A triangle whose sides form a Pythagorean triple is called a **Pythagorean triangle**, and is necessarily a right triangle.

The name is derived from the Pythagorean theorem stating that every right triangle has side lengths satisfying the formula $a^2 + b^2 = c^2$; thus, Pythagorean triples describe the three integer side lengths of a right triangle. However, right triangles with non-integer sides do not form Pythagorean triples. For instance, the triangle with sides *a* = *b* = 1 and *c* = $\sqrt{2}$ is a right triangle, but (1, 1, $\sqrt{2}$) is not a Pythagorean triple because $\sqrt{2}$ is not an integer. Moreover, 1 and $\sqrt{2}$ do not have an integer common multiple because $\sqrt{2}$ is irrational.

Pythagorean triples have been known since ancient times. The oldest known record comes from Plimpton 322, a Babylonian clay tablet from about 1800 BC, written in a sexagesimal number system. It was discovered by Edgar James Bankss shortly after 1900, and sold to George Arthur Plimpton in 1922, for \$10!^[2]



The Pythagorean theorem:
 $a^2 + b^2 = c^2$



Animation demonstrating the simplest Pythagorean triple, $3^2 + 4^2 = 5^2$.

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See also

Notes

References

Examples

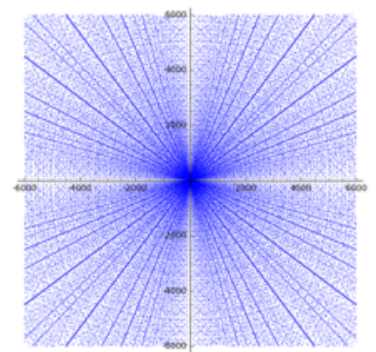
There are 16 primitive Pythagorean triples with $c \leq 100$:

(3, 4, 5)	(5, 12, 13)	(8, 15, 17)	(7, 24, 25)
(20, 21, 29)	(12, 35, 37)	(9, 40, 41)	(28, 45, 53)
(11, 60, 61)	(16, 63, 65)	(33, 56, 65)	(48, 55, 73)
(13, 84, 85)	(36, 77, 85)	(39, 80, 89)	(65, 72, 97)

Note, for example, that (6, 8, 10) is *not* a primitive Pythagorean triple, as it is a multiple of (3, 4, 5). Each of these low- c points forms one of the more easily recognizable radiating lines in the scatter plot.

Additionally these are all the primitive Pythagorean triples with $100 < c \leq 300$:

(20, 99, 101)	(60, 91, 109)	(15, 112, 113)	(44, 117, 125)
(88, 105, 137)	(17, 144, 145)	(24, 143, 145)	(51, 140, 149)
(85, 132, 157)	(119, 120, 169)	(52, 165, 173)	(19, 180, 181)
(57, 176, 185)	(104, 153, 185)	(95, 168, 193)	(28, 195, 197)
(84, 187, 205)	(133, 156, 205)	(21, 220, 221)	(140, 171, 221)
(60, 221, 229)	(105, 208, 233)	(120, 209, 241)	(32, 255, 257)
(23, 264, 265)	(96, 247, 265)	(69, 260, 269)	(115, 252, 277)
(160, 231, 281)	(161, 240, 289)	(68, 285, 293)	



Scatter plot of the legs (a, b) of the first Pythagorean triples with a and b less than 6000. Negative values are included to illustrate the parabolic patterns.

Generating a triple

Euclid's formula^[3] is a fundamental formula for generating Pythagorean triples given an arbitrary pair of integers m and n with $m > n > 0$. The formula states that the integers

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2$$

form a Pythagorean triple. The triple generated by Euclid's formula is primitive if and only if m and n are coprime and not both odd. When both m and n are odd, then a , b , and c will be even, and the triple will not be primitive; however dividing a , b , and c by 2 will yield a primitive triple when m and n are coprime and both odd!^[4]

Every primitive triple arises (after the exchange of a and b , if a is even) from a *unique pair* of coprime numbers m , n , one of which is even. It follows that there are infinitely many primitive Pythagorean triples. This relationship of a , b and c to m and n from Euclid's formula is referenced throughout the rest of this article.

Despite generating all primitive triples, Euclid's formula does not produce all triples—for example, (9, 12, 15) cannot be generated using integer m and n . This can be remedied by inserting an additional parameter k to the formula. The following will generate all Pythagorean triples uniquely:

$$a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad c = k \cdot (m^2 + n^2)$$

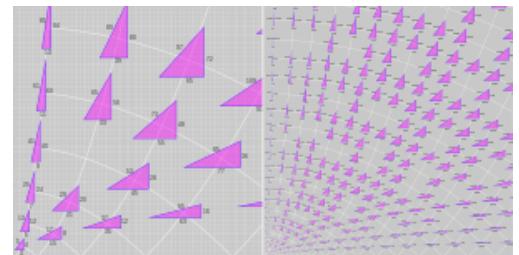
where m , n , and k are positive integers with $m > n$, and with m and n coprime and not both odd.

That these formulas generate Pythagorean triples can be verified by expanding $a^2 + b^2$ using elementary algebra and verifying that the result equals c^2 . Since every Pythagorean triple can be divided through by some integer k to obtain a primitive triple, every triple can be generated uniquely by using the formula with m and n to generate its primitive counterpart and then multiplying through by k as in the last equation.

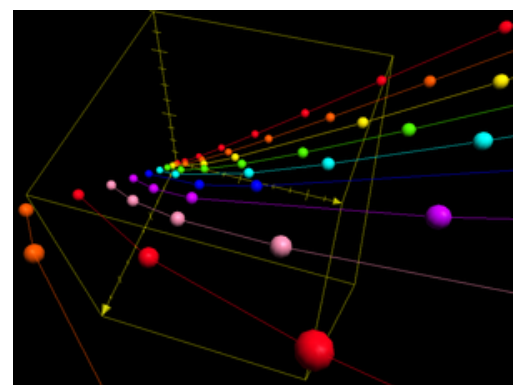
Many formulas for generating triples with particular properties have been developed since the time of Euclid.

Proof of Euclid's formula

That satisfaction of Euclid's formula by a , b , c is sufficient for the triangle to be Pythagorean is apparent from the fact that for positive integers m and n , $m > n$, the a , b , and c given by the formula are all positive integers, and from the fact that



The primitive Pythagorean triples. The odd leg a is plotted on the horizontal axis, the even leg b on the vertical. The curvilinear grid is composed of curves of constant $m - n$ and of constant $m + n$ in Euclid's formula.



A plot of triples generated by Euclid's formula maps out part of the $z^2 = x^2 + y^2$ cone. A constant m or n traces out part of a parabola on the cone.

$$a^2 + b^2 = (m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2 = c^2.$$

A proof of the *necessity* that a , b , c be expressed by Euclid's formula for any primitive Pythagorean triple is as follows.^[5] All such triples can be written as (a, b, c) where $a^2 + b^2 = c^2$ and a, b, c are coprime. Thus a, b, c are pairwise coprime (if a prime number would divide two of them, it would divide also the third one). As a and b are coprime, one is odd, and one may suppose that it is a , by exchanging, if needed, a and b . This implies that b is even and c is odd (if b were odd, c would be even, and c^2 would be a multiple of 4, while $a^2 + b^2$ would be congruent to 2 modulo 4, as an odd square is congruent to 1 modulo 4).

From $a^2 + b^2 = c^2$ we obtain $c^2 - a^2 = b^2$ and hence $(c - a)(c + a) = b^2$. Then $\frac{(c+a)}{b} = \frac{b}{(c-a)}$. Since $\frac{(c+a)}{b}$ is rational, we set it equal to $\frac{m}{n}$ in lowest terms. Thus $\frac{(c-a)}{b} = \frac{n}{m}$, as being the reciprocal of $\frac{(c+a)}{b}$. Then solving

$$\frac{c}{b} + \frac{a}{b} = \frac{m}{n}, \quad \frac{c}{b} - \frac{a}{b} = \frac{n}{m}$$

for $\frac{c}{b}$ and $\frac{a}{b}$ gives

$$\frac{c}{b} = \frac{1}{2} \left(\frac{m}{n} + \frac{n}{m} \right) = \frac{m^2 + n^2}{2mn}, \quad \frac{a}{b} = \frac{1}{2} \left(\frac{m}{n} - \frac{n}{m} \right) = \frac{m^2 - n^2}{2mn}.$$

As $\frac{m}{n}$ is fully reduced, m and n are coprime, and they cannot be both even. If they were both odd, the numerator of $\frac{m^2 - n^2}{2mn}$ would be a multiple of 4 (because an odd square is congruent to 1 modulo 4), and the denominator $2mn$ would not be a multiple of 4, implying that a would be even. Thus one of m and n is odd and the other is even, and the numerators of the two fractions with denominator $2mn$ are odd. Thus these fractions are fully reduced (an odd prime dividing this denominator divides one of m and n and not the other; thus it does not divide $m^2 \pm n^2$). One may thus equate numerators with numerators and denominators with denominators, giving Euclid's formula

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2 \text{ with } m \text{ and } n \text{ coprime and of opposite parities.}$$

A longer but more commonplace proof is given in Maor (2007)^[6] and Sierpiński (2003)^{[7]:4–7}

Interpretation of parameters in Euclid's formula

Suppose the sides of a Pythagorean triangle have lengths $m^2 - n^2$, $2mn$, and $m^2 + n^2$, and suppose the angle between the leg of length $m^2 - n^2$ and the hypotenuse of length $m^2 + n^2$ is denoted as β . Then $\tan \frac{\beta}{2} = \frac{n}{m}$ and the full-angle trigonometric values are $\sin \beta = \frac{2mn}{m^2 + n^2}$, $\cos \beta = \frac{m^2 - n^2}{m^2 + n^2}$, and $\tan \beta = \frac{2mn}{m^2 - n^2}$.^[8]

A variant

The following variant of Euclid's formula is sometimes more convenient, as being closer to symmetric m and n .

If m and n are two odd integers such that $m > n$, then

$$a = mn, \quad b = \frac{m^2 - n^2}{2}, \quad c = \frac{m^2 + n^2}{2}$$

are three integers that form a Pythagorean triple, which is primitive if and only if m and n are coprime. Conversely, every primitive Pythagorean triple arises (after the exchange of a and b , if a is even) from a unique pair $m > n > 0$ of coprime odd integers.

Elementary properties of primitive Pythagorean triples

General properties

The properties of a primitive Pythagorean triple (a, b, c) with $a < b < c$ (without specifying which of a or b is even and which is odd) include:

- $\frac{(c-a)(c-b)}{2}$ is always a perfect square.^[9] As it is only a necessary condition but not a sufficient one, it can be used in checking if a given triple of numbers is *not* a Pythagorean triple when they fail the test. For example, the triple $\{6, 12, 18\}$ passes the test that $(c - a)(c - b)/2$ is a perfect square, but it is not a Pythagorean triple.
- When a triple of numbers a, b and c forms a primitive Pythagorean triple, then $(c$ minus the even leg) and one-half of $(c$ minus the odd leg) are both perfect squares; however this is not a sufficient condition, as the numbers $\{1, 8, 9\}$ pass the perfect squares test but are not a Pythagorean triple since $1^2 + 8^2 \neq 9^2$.
- At most one of a, b, c is a square.^[10]
- The area of a Pythagorean triangle cannot be the square^{[11]:p. 17} or twice the square^{[11]:p. 21} of a natural number
- Exactly one of a, b is odd; c is odd.^[12]
- Exactly one of a, b is divisible by 3.^{[7]:23–25}

- Exactly one of a , b is divisible by 4.^[7]
- Exactly one of a , b , c is divisible by 5.^[7]
- The largest number that always divides abc is 60.^[13]
- All prime factors of c are primes of the form $4n + 1$.^[14] Therefore c is of the form $4n + 1$.
- The area ($K = ab/2$) is a congruent number.^[15] divisible by 6.
- In every Pythagorean triple, the radius of the incircle and the radii of the three excircles are natural numbers. Specifically for a primitive triple the radius of the incircle is $r = n(m - n)$, and the radii of the excircles opposite the sides $m^2 - n^2$, $2mn$, and the hypotenuse $m^2 + n^2$ are respectively $m(m - n)$, $n(m + n)$, and $m(m + n)$.^[16]
- As for any right triangle, the converse of Hales' theorem says that the diameter of the circumcircle equals the hypotenuse; hence for primitive triples the circumdiameter is $m^2 + n^2$, and the circumradius is half of this and thus is rational but non-integer (since m and n have opposite parity).
- When the area of a Pythagorean triangle is multiplied by the curvatures of its incircle and 3 excircles, the result is four positive integers $w > x > y > z$, respectively. Integers $-w$, x , y , z satisfy Descartes's Circle Equation.^[17] Equivalently, the radius of the outer Soddy circle of any right triangle is equal to its semiperimeter. The outer Soddy center is located at D , where $ACBD$ is a rectangle, ACB the right triangle and AB its hypotenuse.^{[17]:p. 6}
- There are no Pythagorean triples in which the hypotenuse and one leg are the legs of another Pythagorean triple; this is one of the equivalent forms of Fermat's right triangle theorem.^{[11]:p. 14}
- Each primitive Pythagorean triangle has a ratio of area to square of semiperimeter that is unique to itself and is given by^[18]

$$\frac{K}{s^2} = \frac{n(m-n)}{m(m+n)} = 1 - \frac{c}{s}.$$

- No primitive Pythagorean triangle has an integer altitude from the hypotenuse; that is, every primitive Pythagorean triangle is indecomposable.^[19]
- The set of all primitive Pythagorean triples forms a rooted ternary tree in a natural way; see Tree of primitive Pythagorean triples
- Neither of the acute angles of a Pythagorean triangle can be a rational number of degrees.^[20] (This follows from Niven's theorem)

Special cases

In addition, special Pythagorean triples with certain additional properties can be guaranteed to exist:

- Every integer greater than 2 that is not congruent to 2 mod 4 (in other words, every integer greater than 2 which is not of the form $4n + 2$) is part of a primitive Pythagorean triple.
- Every integer greater than 2 is part of a primitive or non-primitive Pythagorean triple. For example, the integers 6, 10, 14, and 18 are not part of primitive triples, but are part of the non-primitive triples (6, 8, 10), (14, 48, 50) and (18, 80, 82).
- There exist infinitely many Pythagorean triples in which the hypotenuse and the longer of the two legs differ by exactly one (such triples are necessarily primitive). One method to generate such triples is the relation $(2j + 1)^2 + [2n(n + 1)]^2 = [2n(n + 1) + 1]^2$, leading to triples (3,4,5), (5,12,13), (7,24,25), etc. More generally for every odd integer j , there exist infinitely many primitive Pythagorean triples in which the hypotenuse and the even leg differ by j^2 .
- There exist infinitely many primitive Pythagorean triples in which the hypotenuse and the longer of the two legs differ by exactly two. Generalization: For every integer $k > 0$, there exist infinitely many primitive Pythagorean triples in which the hypotenuse and the odd leg differ by $2k^2$.
- There exist infinitely many Pythagorean triples in which the two legs differ by exactly one. For example, $20^2 + 21^2 = 29^2$; these are generated by Euclid's formula when $\frac{m-n}{n}$ is a convergent to $\sqrt{2}$.
- For each natural number n , there exist n Pythagorean triples with different hypotenuses and the same area.
- For each natural number n , there exist at least n different Pythagorean triples with the same a , where a is some natural number
- For each natural number n , there exist at least n different Pythagorean triples with the same hypotenuse.^{[7]:31}
- There exist infinitely many Pythagorean triples with square numbers for both the hypotenuse and the sum of the legs $a + b$. According to Fermat, the **smallest** such triple^[21] has sides $a = 4,565,486,027,761$; $b = 1,061,652,293,520$; and $c = 4,687,298,610,289$. Here $a + b = 2,372,159^2$ and $c = 2,165,017^2$. This is generated by Euclid's formula with parameter values $m = 2,150,905$ and $n = 246,792$.
- There exist non-primitive Pythagorean triangles with integer altitude from the hypotenuse.^{[22][23]} Such Pythagorean triangles are known as decomposable since they can be split along this altitude into two separate and smaller Pythagorean triangles.^[19]

Geometry of Euclid's formula

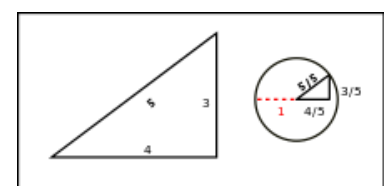
Rational points on a unit circle

Euclid's formula for a Pythagorean triple

$$a = 2mn, \quad b = m^2 - n^2, \quad c = m^2 + n^2$$

can be understood in terms of the geometry of rational points on the unit circle (Trautman 1998).

In fact, a point in the Cartesian plane with coordinates (x, y) belongs to the unit circle if $x^2 + y^2 = 1$. The point is *rational* if x and y are rational numbers that is, if there are coprime integers a , b , c such that



3,4,5 maps to x,y point (4/5,3/5) on the unit circle

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

By multiplying both members by c^2 , one can see that the rational points on the circle are in one-to-one correspondence with the primitive Pythagorean triples.

The unit circle may also be defined by a parametric equation

$$x = \frac{1-t^2}{1+t^2} \quad y = \frac{2t}{1+t^2}.$$

Euclid's formula for Pythagorean triples means that, except for $(-1, 0)$, a point on the circle is rational if and only if the corresponding value of t is a rational number.

Stereographic approach

There is therefore a correspondence between points on the unit circle with rational coordinates and primitive Pythagorean triples. At this point, Euclid's formulae can be derived either by methods of trigonometry or equivalently by using the stereographic projection

For the stereographic approach, suppose that P' is a point on the x -axis with rational coordinates

$$P' = \left(\frac{m}{n}, 0\right).$$

Then, it can be shown by basic algebra that the point P has coordinates

$$P = \left(\frac{2\left(\frac{m}{n}\right)}{\left(\frac{m}{n}\right)^2 + 1}, \frac{\left(\frac{m}{n}\right)^2 - 1}{\left(\frac{m}{n}\right)^2 + 1}\right) = \left(\frac{2mn}{m^2 + n^2}, \frac{m^2 - n^2}{m^2 + n^2}\right).$$

This establishes that each rational point of the x -axis goes over to a rational point of the unit circle. The converse, that every rational point of the unit circle comes from such a point of the x -axis, follows by applying the inverse stereographic projection. Suppose that $P(x, y)$ is a point of the unit circle with x and y rational numbers. Then the point P' obtained by stereographic projection onto the x -axis has coordinates

$$\left(\frac{x}{1-y}, 0\right)$$

which is rational.

In terms of algebraic geometry, the algebraic variety of rational points on the unit circle is birational to the affine line over the rational numbers. The unit circle is thus called a rational curve, and it is this fact which enables an explicit parameterization of the (rational number) points on it by means of rational functions.

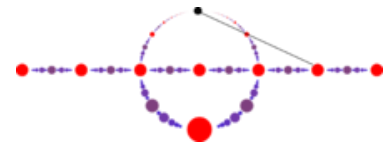
Pythagorean triangles in a 2D lattice

A 2D lattice is a regular array of isolated points where if any one point is chosen as the Cartesian origin $(0, 0)$, then all the other points are at (x, y) where x and y range over all positive and negative integers. Any Pythagorean triangle with triple (a, b, c) can be drawn within a 2D lattice with vertices at coordinates $(0, 0)$, $(a, 0)$ and $(0, b)$. The count of lattice points lying strictly within the bounds of the triangle is given by $\frac{(a-1)(b-1) - \gcd(a,b) + 1}{2}$; [24] for primitive Pythagorean triples this interior lattice count is $\frac{(a-1)(b-1)}{2}$. The area (by Pick's theorem equal to one less than the interior lattice count plus half the boundary lattice count) equals $\frac{ab}{2}$.

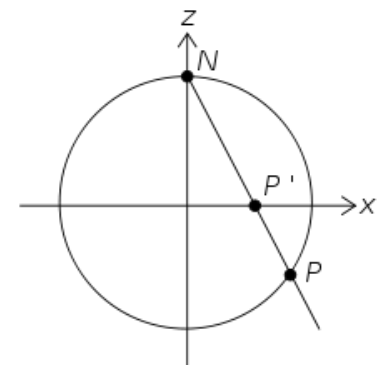
The first occurrence of two primitive Pythagorean triples sharing the same area occurs with triangles with sides $(20, 21, 29)$, $(12, 35, 37)$ and common area 210 (sequence A093536 in the OEIS). The first occurrence of two primitive Pythagorean triples sharing the same interior lattice count occurs with $(18108, 252685, 253333)$, $(28077, 162964, 165365)$ and interior lattice count 2287674594 (sequence A225760 in the OEIS). Three primitive Pythagorean triples have been found sharing the same area: $(4485, 5852, 7373)$, $(3059, 8580, 9109)$, $(1380, 19019, 19069)$ with area 13123110. As yet, no set of three primitive Pythagorean triples have been found sharing the same interior lattice count.

Enumeration of primitive Pythagorean triples

By Euclid's formula all primitive Pythagorean triples can be generated from integers m and n with $m > n > 0$, $m + n$ odd and $\gcd(m, n) = 1$. Hence there is a 1 to 1 mapping of rationals (in lowest terms) to primitive Pythagorean triples where $\frac{2}{m}$ is in the interval $(0, 1)$ and $m + n$ odd.



The rational points on a circle correspond, under stereographic projection, to the rational points of the line.



Stereographic projection of the unit circle onto the x -axis. Given a point P on the unit circle, draw a line from P to the point $N = (0, 1)$ (the *north pole*). The point P' where the line intersects the x -axis is the stereographic projection of P . Inversely, starting with a point P' on the x -axis, and drawing a line from P' to N , the inverse stereographic projection is the point P where the line intersects the unit circle.

The reverse mapping from a primitive triple (a, b, c) where $c > b > a > 0$ to a rational $\frac{n}{m}$ is achieved by studying the two sums $a + c$ and $b + c$. One of these sums will be a square that can be equated to $(m + n)^2$ and the other will be twice a square that can be equated to $2m^2$. It is then possible to determine the rational $\frac{n}{m}$.

In order to enumerate primitive Pythagorean triples the rational can be expressed as an ordered pair (n, m) and mapped to an integer using a pairing function such as Cantor's pairing function. An example can be seen at (sequence A277557 in the OEIS). It begins

8, 18, 19, 32, 33, 34, . . . and gives rationals
 $\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}, \dots$ these, in turn, generate primitive triples
(3, 4, 5), (5, 12, 13), (8, 15, 17), (7, 24, 25), (20, 21, 29), (12, 35, 37), . . .

Spinors and the modular group

Pythagorean triples can likewise be encoded into a matrix of the form

$$X = \begin{bmatrix} c + b & a \\ a & c - b \end{bmatrix}.$$

A matrix of this form is isymmetric. Furthermore, the determinant of X is

$$\det X = c^2 - a^2 - b^2$$

which is zero precisely when (a,b,c) is a Pythagorean triple. If X corresponds to a Pythagorean triple, then as a matrix it must have rank 1.

Since X is symmetric, it follows from a result in linear algebra that there is a column vector $\xi = [m \ n]^T$ such that the outer product

$$X = 2 \begin{bmatrix} m \\ n \end{bmatrix} [m \ n] = 2 \xi \xi^T \tag{1}$$

holds, where the T denotes the matrix transpose. The vector ξ is called a spinor (for the Lorentz group $SO(1, 2)$). In abstract terms, the Euclid formula means that each primitive Pythagorean triple can be written as the outer product with itself of a spinor with integer entries, as in (1).

The modular group Γ is the set of 2×2 matrices with integer entries

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

with determinant equal to one: $\alpha\delta - \beta\gamma = 1$. This set forms a group, since the inverse of a matrix in Γ is again in Γ , as is the product of two matrices in Γ . The modular group acts on the collection of all integer spinors. Furthermore, the group is transitive on the collection of integer spinors with relatively prime entries. For if $[m \ n]^T$ has relatively prime entries, then

$$\begin{bmatrix} m & -v \\ n & u \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$$

where u and v are selected (by the Euclidean algorithm) so that $mu + nv = 1$.

By acting on the spinor ξ in (1), the action of Γ goes over to an action on Pythagorean triples, provided one allows for triples with possibly negative components. Thus if A is a matrix in Γ , then

$$2(A\xi)(A\xi)^T = AXA^T \tag{2}$$

gives rise to an action on the matrix X in (1). This does not give a well-defined action on primitive triples, since it may take a primitive triple to an imprimitive one. It is convenient at this point (per Trautman 1998) to call a triple (a,b,c) **standard** if $c > 0$ and either (a,b,c) are relatively prime or $(a/2,b/2,c/2)$ are relatively prime with $a/2$ odd. If the spinor $[m \ n]^T$ has relatively prime entries, then the associated triple (a,b,c) determined by (1) is a standard triple. It follows that the action of the modular group is transitive on the set of standard triples.

Alternatively, restrict attention to those values of m and n for which m is odd and n is even. Let the subgroup $\Gamma(2)$ of Γ be the kernel of the group homomorphism

$$\Gamma = \text{SL}(2, \mathbf{Z}) \rightarrow \text{SL}(2, \mathbf{Z}_2)$$

where $\text{SL}(2, \mathbf{Z}_2)$ is the special linear group over the finite field $\mathbf{Z}_2$ of integers modulo 2. Then $\Gamma(2)$ is the group of unimodular transformations which preserve the parity of each entry. Thus if the first entry of ξ is odd and the second entry is even, then the same is true of $A\xi$ for all $A \in \Gamma(2)$. In fact, under the action (2), the group $\Gamma(2)$ acts transitively on the collection of primitive Pythagorean triples (Alperin 2005).

The group $\Gamma(2)$ is the free group whose generators are the matrices

$$U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Consequently, every primitive Pythagorean triple can be obtained in a unique way as a product of copies of the matrices U and L .

Parent/child relationships

By a result of Berggren (1934), all primitive Pythagorean triples can be generated from the (3, 4, 5) triangle by using the three linear transformations T_1 , T_2 , T_3 below, where a , b , c are sides of a triple:

	new side a	new side b	new side c
T_1 :	$a - 2b + 2c$	$2a - b + 2c$	$2a - 2b + 3c$
T_2 :	$a + 2b + 2c$	$2a + b + 2c$	$2a + 2b + 3c$
T_3 :	$-a + 2b + 2c$	$-2a + b + 2c$	$-2a + 2b + 3c$

In other words, every primitive triple will be a "parent" to three additional primitive triples. Starting from the initial node with $a = 3$, $b = 4$, and $c = 5$, the operation T_1 produces the new triple

$$(3 - (2 \times 4) + (2 \times 5), (2 \times 3) - 4 + (2 \times 5), (2 \times 3) - (2 \times 4) + (3 \times 5)) = (5, 12, 13),$$

and similarly T_2 and T_3 produce the triples (21, 20, 29) and (15, 8, 17).

The linear transformations T_1 , T_2 , and T_3 have a geometric interpretation in the language of quadratic forms. They are closely related to (but are not equal to) reflections generating the orthogonal group of $x^2 + y^2 - z^2$ over the integers.^[25]

Relation to Gaussian integers

Alternatively, Euclid's formulae can be analyzed and proven using the Gaussian integers.^[26] Gaussian integers are complex numbers of the form $\alpha = u + vi$, where u and v are ordinary integers and i is the square root of negative one. The units of Gaussian integers are ± 1 and $\pm i$. The ordinary integers are called the rational integers and denoted as $\mathbf{Z}$. The Gaussian integers are denoted as $\mathbf{Z}[i]$. The right-hand side of the Pythagorean theorem may be factored in Gaussian integers:

$$c^2 = a^2 + b^2 = (a + bi)\overline{(a + bi)} = (a + bi)(a - bi).$$

A primitive Pythagorean triple is one in which a and b are coprime, i.e., they share no prime factors in the integers. For such a triple, either a or b is even, and the other is odd; from this, it follows that c is also odd.

The two factors $z := a + bi$ and $z^* := a - bi$ of a primitive Pythagorean triple each equal the square of a Gaussian integer. This can be proved using the property that every Gaussian integer can be factored uniquely into Gaussian primes up to units.^[27] (This unique factorization follows from the fact that, roughly speaking, a version of the Euclidean algorithm can be defined on them.) The proof has three steps. First, if a and b share no prime factors in the integers, then they also share no prime factors in the Gaussian integers. (Assume $a = gu$ and $b = gv$ with Gaussian integers g , u and v and g not a unit. Then u and v lie on the same line through the origin. All Gaussian integers on such a line are integer multiples of some Gaussian integer h . But then the integer $gh \neq \pm 1$ divides both a and b .) Second, it follows that z and z^* likewise share no prime factors in the Gaussian integers. For if they did, then their common divisor δ would also divide $z + z^* = 2a$ and $z - z^* = 2ib$. Since a and b are coprime, that implies that δ divides $2 = (1 + i)(1 - i) = i(1 - i)^2$. From the formula $c^2 = zz^*$, that in turn would imply that c is even, contrary to the hypothesis of a primitive Pythagorean triple. Third, since c^2 is a square, every Gaussian prime in its factorization is doubled, i.e., appears an even number of times. Since z and z^* share no prime factors, this doubling is also true for them. Hence z and z^* are squares.

Thus, the first factor can be written

$$a + bi = \varepsilon(m + ni)^2, \quad \varepsilon \in \{\pm 1, \pm i\}.$$

The real and imaginary parts of this equation give the two formulas:

$$\begin{cases} \varepsilon = +1, & a = +(m^2 - n^2), & b = +2mn; \\ \varepsilon = -1, & a = -(m^2 - n^2), & b = -2mn; \\ \varepsilon = +i, & a = -2mn, & b = +(m^2 - n^2); \\ \varepsilon = -i, & a = +2mn, & b = -(m^2 - n^2). \end{cases}$$

For any primitive Pythagorean triple, there must be integers m and n such that these two equations are satisfied. Hence, every Pythagorean triple can be generated from some choice of these integers.

As perfect square Gaussian integers

If we consider the square of a Gaussian integer we get the following direct interpretation of Euclid's formulae as representing a perfect square Gaussian integers

$$(m + ni)^2 = (m^2 - n^2) + 2mni.$$

Using the facts that the Gaussian integers are a Euclidean domain and that for a Gaussian integer p $|p|^2$ is always a square it is possible to show that a Pythagorean triples correspond to the square of a prime Gaussian integer if the hypotenuse is prime.

If the Gaussian integer is not prime then it is the product of two Gaussian integers p and q with $|p|^2$ and $|q|^2$ integers. Since magnitudes multiply in the Gaussian integers, the product must be $|p||q|$, which when squared to find a Pythagorean triple must be composite. The contrapositive completes the proof.

Relation to ellipses with integral dimensions

With reference to the figure and the definition of the foci of an ellipse, F_1 and F_2 , for any point P on the ellipse, $F_1P + PF_2$ is constant.

As points A and B are both on the ellipse, $F_1A + AF_2 = F_1B + BF_2$. Due to symmetry, $F_1A + AF_2 = F_2A' + AF_2 = AA' = 2 AC$, and $F_1B + BF_2 = 2 BF_2$. Hence, $AC = BF_2$.

Thus, if BCF_2 is a right-angle triangle with integral sides, the separation of the foci, linear eccentricity, minor axis and major axis are all also integers.^[28]

Distribution of triples

There are a number of results on the distribution of Pythagorean triples. In the scatter plot, a number of obvious patterns are already apparent. Whenever the legs (a,b) of a primitive triple appear in the plot, all integer multiples of (a,b) must also appear in the plot, and this property produces the appearance of lines radiating from the origin in the diagram.

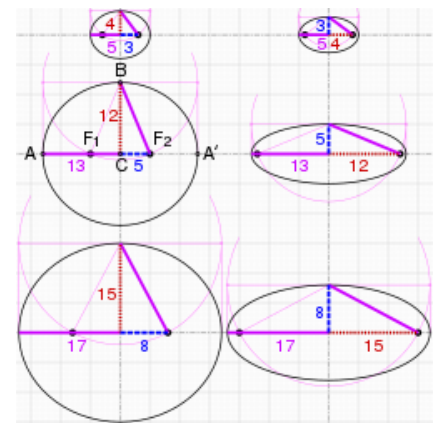
Within the scatter, there are sets of parabolic patterns with a high density of points and all their foci at the origin, opening up in all four directions. Different parabolas intersect at the axes and appear to reflect off the axis with an incidence angle of 45 degrees, with a third parabola entering in a perpendicular fashion. Within this quadrant, each arc centered on the origin shows that section of the parabola that lies between its tip and its intersection with its semi-latus rectum.

These patterns can be explained as follows. If $a^2/4n$ is an integer, then $(a, |n - a^2/4n|, n + a^2/4n)$ is a Pythagorean triple. (In fact every Pythagorean triple (a, b, c) can be written in this way with integer n , possibly after exchanging a and b , since $n = (b + c)/2$ and a and b cannot both be odd.) The Pythagorean triples thus lie on curves given by $b = |n - a^2/4n|$, that is, parabolas reflected at the a -axis, and the corresponding curves with a and b interchanged. If a is varied for a given n (i.e. on a given parabola), integer values of b occur relatively frequently if n is a square or a small multiple of a square. If several such values happen to lie close together, the corresponding parabolas approximately coincide, and the triples cluster in a narrow parabolic strip. For instance, $38^2 = 1444$, $2 \times 27^2 = 1458$, $3 \times 22^2 = 1452$, $5 \times 17^2 = 1445$ and $10 \times 12^2 = 1440$; the corresponding parabolic strip around $n \approx 1450$ is clearly visible in the scatter plot.

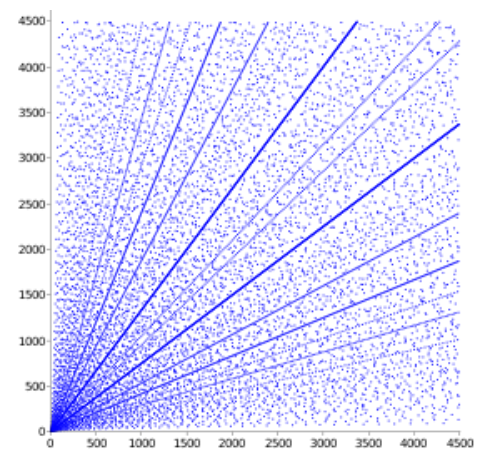
The angular properties described above follow immediately from the functional form of the parabolas. The parabolas are reflected at the a -axis at $a = 2n$, and the derivative of b with respect to a at this point is -1 ; hence the incidence angle is 45° . Since the clusters, like all triples, are repeated at integer multiples, the value $2n$ also corresponds to a cluster. The corresponding parabola intersects the b -axis at right angles at $b = 2n$, and hence its reflection upon interchange of a and b intersects the a -axis at right angles at $a = 2n$, precisely where the parabola for n is reflected at the a -axis. (The same is of course true for a and b interchanged.)

Albert Fässler and others provide insights into the significance of these parabolas in the context of conformal mappings.^{[29][30]}

Special cases



Relationship between Pythagorean triples and ellipses with integral linear eccentricity, and major and minor axes, for the first 3 Pythagorean triples



A scatter plot of the legs (a,b) of the first Pythagorean triples with a and b less than 4500.

The Platonic sequence

The case $n = 1$ of the more general construction of Pythagorean triples has been known for a long time. Proclus, in his commentary to the 47th Proposition of the first book of Euclid's Elements, describes it as follows:

Certain methods for the discovery of triangles of this kind are handed down, one which they refer to Plato, and another to Pythagoras. (The latter) starts from odd numbers. For it makes the odd number the smaller of the sides about the right angle; then it takes the square of it, subtracts unity and makes half the difference the greater of the sides about the right angle; lastly it adds unity to this and so forms the remaining side, the hypotenuse.

...For the method of Plato argues from even numbers. It takes the given even number and makes it one of the sides about the right angle; then, bisecting this number and squaring the half, it adds unity to the square to form the hypotenuse, and subtracts unity from the square to form the other side about the right angle. ... Thus it has formed the same triangle that which was obtained by the other method.

In equation form, this becomes:

a is odd (Pythagoras, c. 540 BC):

$$\text{side } a : \text{side } b = \frac{a^2 - 1}{2} : \text{side } c = \frac{a^2 + 1}{2}.$$

a is even (Plato, c. 380 BC):

$$\text{side } a : \text{side } b = \left(\frac{a}{2}\right)^2 - 1 : \text{side } c = \left(\frac{a}{2}\right)^2 + 1$$

It can be shown that all Pythagorean triples can be obtained, with appropriate rescaling, from the basic Platonic sequence $(a, (a^2 - 1)/2$ and $(a^2 + 1)/2)$ by allowing a to take non-integer rational values. If a is replaced with the fraction m/n in the sequence, the result is equal to the 'standard' triple generator $(2mn, m^2 - n^2, m^2 + n^2)$ after rescaling. It follows that every triple has a corresponding rational a value which can be used to generate a similar triangle (one with the same three angles and with sides in the same proportions as the original). For example, the Platonic equivalent of (56, 33, 65) is generated by $a = m/n = 7/4$ as $(a, (a^2 - 1)/2, (a^2 + 1)/2) = (56/32, 33/32, 65/32)$. The Platonic sequence itself can be derived by following the steps for 'splitting the square' described in Diophantus II.VIII.

The Jacobi–Madden equation

The equation,

$$a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$$

is equivalent to the special Pythagorean triple,

$$(a^2 + ab + b^2)^2 + (c^2 + cd + d^2)^2 = ((a + b)^2 + (a + b)(c + d) + (c + d)^2)^2$$

There is an infinite number of solutions to this equation as solving for the variables involves an elliptic curve. Small ones are,

$$a, b, c, d = -2634, 955, 1770, 5400$$

$$a, b, c, d = -31764, 7590, 27385, 48150$$

Equal sums of two squares

One way to generate solutions to $a^2 + b^2 = c^2 + d^2$ is to parametrize a, b, c, d in terms of integers m, n, p, q as follows:^[31]

$$(m^2 + n^2)(p^2 + q^2) = (mp - nq)^2 + (np + mq)^2 = (mp + nq)^2 + (np - mq)^2.$$

Equal sums of two fourth powers

Given two sets of Pythagorean triples,

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2$$

$$(c^2 - d^2)^2 + (2cd)^2 = (c^2 + d^2)^2$$

the problem of finding equal products of a non-hypotenuse side and the hypotenuse,

$$(a^2 - b^2)(a^2 + b^2) = (c^2 - d^2)(c^2 + d^2)$$

is easily seen to be equivalent to the equation,

$$a^4 - b^4 = c^4 - d^4$$

and was first solved by Euler as $a, b, c, d = 133, 59, 158, 134$. Since he showed this is a rational point in an elliptic curve, then there is an infinite number of solutions. In fact, he also found a 7th degree polynomial parameterization.

Descartes' Circle Theorem

For the case of Descartes' circle theorem where all variables are squares,

$$2(a^4 + b^4 + c^4 + d^4) = (a^2 + b^2 + c^2 + d^2)^2$$

Euler showed this is equivalent to three simultaneous Pythagorean triples,

$$(2ab)^2 + (2cd)^2 = (a^2 + b^2 - c^2 - d^2)^2$$

$$(2ac)^2 + (2bd)^2 = (a^2 - b^2 + c^2 - d^2)^2$$

$$(2ad)^2 + (2bc)^2 = (a^2 - b^2 - c^2 + d^2)^2$$

There is also an infinite number of solutions, and for the special case when $a + b = c$, then the equation simplifies to,

$$4(a^2 + ab + b^2) = d^2$$

with small solutions as $a, b, c, d = 3, 5, 8, 14$ and can be solved as binary quadratic forms

Almost-isosceles Pythagorean triples

No Pythagorean triples are isosceles, because the ratio of the hypotenuse to either other side is $\sqrt{2}$, but $\sqrt{2}$ cannot be expressed as the ratio of 2 integers

There are, however, right-angled triangles with integral sides for which the lengths of the non-hypotenuse sides differ by one, such as,

$$3^2 + 4^2 = 5^2$$

$$20^2 + 21^2 = 29^2$$

and an infinite number of others. They can be completely parameterized as,

$$\left(\frac{x-1}{2}\right)^2 + \left(\frac{x+1}{2}\right)^2 = y^2$$

where $\{x, y\}$ are the solutions to the Pell equation $x^2 - 2y^2 = -1$.

When it is the longer non-hypotenuse side and hypotenuse that differ by one, such as in

$$5^2 + 12^2 = 13^2$$

$$7^2 + 24^2 = 25^2$$

then the complete solution is

$$(2m+1)^2 + (2m^2+2m)^2 = (2m^2+2m+1)^2$$

which also shows that all odd numbers (greater than 1) appear in a primitive Pythagorean triple.

Fibonacci numbers in Pythagorean triples

Starting with 5, every second Fibonacci number is the length of the hypotenuse of a right triangle with integer sides, or in other words, the largest number in a Pythagorean triple. The length of the longer leg of this triangle is equal to the sum of the three sides of the preceding triangle in this series of triangles, and the shorter leg is equal to the difference between the preceding bypassed Fibonacci number and the shorter leg of the preceding triangle.

Generalizations

There are several ways to generalize the concept of Pythagorean triples.

Pythagorean quadruple

A set of four positive integers a , b , c and d such that $a^2 + b^2 + c^2 = d^2$ is called a Pythagorean quadruple. The simplest example is (1, 2, 2, 3), since $1^2 + 2^2 + 2^2 = 3^2$. The next simplest (primitive) example is (2, 3, 6, 7), since $2^2 + 3^2 + 6^2 = 7^2$.

All quadruples are given by the formula

$$(m^2 + n^2 - p^2 - q^2)^2 + (2mq + 2np)^2 + (2nq - 2mp)^2 = (m^2 + n^2 + p^2 + q^2)^2.$$

Pythagorean n -tuple

Using the simple algebraic identity,

$$(x_1^2 - x_0)^2 + (2x_1)^2 x_0 = (x_1^2 + x_0)^2$$

for arbitrary x_0, x_1 , it is easy to prove that the square of the sum of n squares is itself the sum of n squares by letting $x_0 = x_2^2 + x_3^2 + \dots + x_n^2$ and then distributing terms.^[32] One can see how Pythagorean triples and quadruples are just the particular cases $x_0 = x_2^2$ and $x_0 = x_2^2 + x_3^2$, respectively, and so on for other n , with quintuples given by

$$(a^2 - b^2 - c^2 - d^2)^2 + (2ab)^2 + (2ac)^2 + (2ad)^2 = (a^2 + b^2 + c^2 + d^2)^2.$$

Since the sum $F(k, m)$ of k consecutive squares beginning with m^2 is given by the formula^[33]

$$F(k, m) = km(k - 1 + m) + \frac{k(k - 1)(2k - 1)}{6}$$

one may find values (k, m) so that $F(k, m)$ is a square, such as one by Hirschhorn where the number of terms is itself a square^[34]

$$m = \frac{v^4 - 24v^2 - 25}{48}, \quad k = v^2, \quad F(m, k) = \frac{v^5 + 47v}{48}$$

and $v \geq 5$ is any integer not divisible by 2 or 3. For the smallest case $v = 5$, hence $k = 25$, this yields the well-known cannonball-stacking problem of Lucas,

$$0^2 + 1^2 + 2^2 + \dots + 24^2 = 70^2$$

a fact which is connected to the Leech lattice

In addition, if in a Pythagorean n -tuple ($n \geq 4$) all addends are consecutive except one, one can use the equation^[35]

$$F(k, m) + p^2 = (p + 1)^2$$

Since the second power of p cancels out, this is only linear and easily solved for as $p = \frac{F(k, m) - 1}{2}$ though k, m should be chosen so that p is an integer, with a small example being $k = 5, m = 1$ yielding,

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 27^2 = 28^2$$

Thus, one way of generating Pythagorean n -tuples is by using, for various x ,^[36]

$$x^2 + (x + 1)^2 + \dots + (x + q)^2 + p^2 = (p + 1)^2,$$

where $q = n - 2$ and where

$$p = \frac{(q + 1)x^2 + q(q + 1)x + \frac{q(q + 1)(2q + 1)}{6} - 1}{2}.$$

Fermat's Last Theorem

A generalization of the concept of Pythagorean triples is the search for triples of positive integers a , b , and c , such that $a^n + b^n = c^n$, for some n strictly greater than 2. Pierre de Fermat in 1637 claimed that no such triple exists, a claim that came to be known as Fermat's Last Theorem because it took longer than any other conjecture by Fermat to be proven or disproven. The first proof was given by Andrew Wiles in 1994.

$n - 1$ or n n th powers summing to n th power

Another generalization is searching for sequences of $n + 1$ positive integers for which the n th power of the last is the sum of the n th powers of the previous terms. The smallest sequences for known values of n are:

- $n = 3$: {3, 4, 5; 6}.
- $n = 4$: {30, 120, 272, 315; 353}
- $n = 5$: {19, 43, 46, 47, 67; 72}
- $n = 7$: {127, 258, 266, 413, 430, 439, 525; 568}
- $n = 8$: {90, 223, 478, 524, 748, 1088, 1190, 1324; 1409}

For the $n=3$ case, in which $x^3 + y^3 + z^3 = w^3$, called the [Fermat cubic](#), a general formula exists giving all solutions.

A slightly different generalization allows the sum of $(k + 1)$ n th powers to equal the sum of $(n - k)$ n th powers. For example:

- ($n = 3$): $1^3 + 12^3 = 9^3 + 10^3$, made famous by Hardy's recollection of a conversation with [Ramanujan](#) about the number 1729 being the smallest number that can be expressed as a sum of two cubes in two distinct ways.

There can also exist $n - 1$ positive integers whose n th powers sum to n th power (though, by [Fermat's last theorem](#), not for $n = 3$); these are counterexamples to [Euler's sum of powers conjecture](#). The smallest known counterexamples are^{[37][38][13]}

- $n = 4$: (95800, 217519, 414560; 422481)
- $n = 5$: (27, 84, 110, 133; 144)

Heronian triangle triples

A **Heronian triangle** is commonly defined as one with integer sides whose area is also an integer, and we shall consider Heronian triangles with *distinct* integer sides. The lengths of the sides of such a triangle form a **Heronian triple** (a, b, c) provided $a < b < c$. Clearly, any Pythagorean triple is a Heronian triple, since in a Pythagorean triple at least one of the legs a, b must be even, so that the area $ab/2$ is an integer. Not every Heronian triple is a Pythagorean triple, however, as the example (4, 13, 15) with area 24 shows.

If (a, b, c) is a Heronian triple, so is (ma, mb, mc) where m is any positive integer greater than one. The Heronian triple (a, b, c) is **primitive** provided a, b, c are pairwise relatively prime (as with a Pythagorean triple). Here are a few of the simplest primitive Heronian triples that are not Pythagorean triples:

(4, 13, 15) with area 24
(3, 25, 26) with area 36
(7, 15, 20) with area 42
(6, 25, 29) with area 60
(11, 13, 20) with area 66
(13, 14, 15) with area 84
(13, 20, 21) with area 126

By [Heron's formula](#), the extra condition for a triple of positive integers (a, b, c) with $a < b < c$ to be Heronian is that

$$(a^2 + b^2 + c^2)^2 - 2(a^4 + b^4 + c^4)$$

or equivalently

$$2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)$$

be a nonzero perfect square divisible by 16.

Application to cryptography

Primitive Pythagorean triples have been used in cryptography as random sequences and for the generation of keys.^[39]

See also

- | | | |
|---|--|--|
| ▪ Boolean Pythagorean triples problem | ▪ Heronian triangle | ▪ Plimpton 322 |
| ▪ Congruum | ▪ Hilbert's theorem 90 | ▪ Pythagorean prime |
| ▪ Diophantus II.VIII | ▪ Integer triangle | ▪ Pythagorean quadruple |
| ▪ Eisenstein triple | ▪ Modular arithmetic | ▪ Tangent half-angle formula |
| ▪ Euler brick | ▪ Nonhypotenuse number | ▪ Trigonometric identity |

Notes

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- The Trinary Tree(s) underlying Primitive Pythagorean Triples at cut-the-knot
- Theoretical properties of the Pythagorean Triples and connections to geometry
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