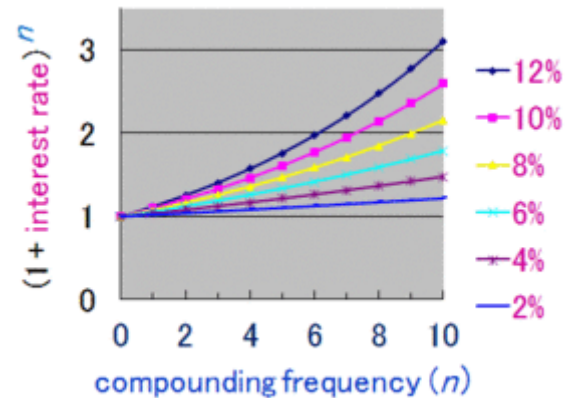


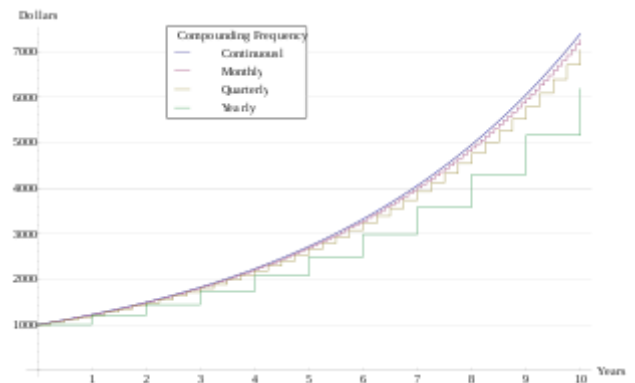
Compound interest

Compound interest is the addition of interest to the principal sum of a loan or deposit, or in other words, interest on interest. It is the result of reinvesting interest, rather than paying it out, so that interest in the next period is then earned on the principal sum plus previously accumulated interest. Compound interest is standard in finance and economics.

Compound interest may be contrasted with *simple interest*, where interest is not added to the principal, so there is no compounding. The *simple annual interest rate* is the interest amount per period, multiplied by the number of periods per year. The simple annual interest rate is also known as the **nominal interest rate** (not to be confused with the interest rate not adjusted for inflation, which goes by the same name).



Effective interest rates



The effect of earning 20% annual interest on an initial \$1,000 investment at various compounding frequencies

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Compounding frequency

The *compounding frequency* is the number of times per year (or other unit of time) the accumulated interest is paid out, or *capitalized* (credited to the account), on a regular basis. The frequency could be yearly, half-yearly, quarterly, monthly, weekly, daily, or continuously (or not at all, until maturity).

For example, monthly capitalization with annual rate of interest means that the compounding frequency is 12, with time periods measured in months.

The effect of compounding depends on:

1. The nominal interest rate which is applied and
2. The frequency interest is compounded.

Compound interest calculator is a tool that allows to calculate compounding interest on loans or investments as well as help to figure out what opening balance is needed to achieve a specific investment goal.^[1]

Annual equivalent rate

The nominal rate cannot be directly compared between loans with different compounding frequencies. Both the nominal interest rate and the compounding frequency are required in order to compare interest-bearing financial instruments.

To assist consumers compare retail financial products more fairly and easily, many countries require financial institutions to disclose the annual compound interest rate on deposits or advances on a comparable basis. The interest rate on an annual equivalent basis may be referred to variously in different markets as annual percentage rate (APR), annual equivalent rate (AER), effective interest rate, effective annual rate, annual percentage yield and other terms. The effective annual rate is the total accumulated interest that would be payable up to the end of one year divided by the principal sum.

There are usually two aspects to the rules defining these rates:

1. The rate is the annualised compound interest rate, and
2. There may be charges other than interest. The effect of fees or taxes which the customer is charged, and which are directly related to the product, may be included. Exactly which fees and taxes are included or excluded varies by country, may or may not be comparable between different jurisdictions, because the use of such terms may be inconsistent, and vary according to local practice.

Examples

- 1,000 Brazilian real (BRL) is deposited into a Brazilian savings account paying 20% per annum, compounded annually. At the end of one year $1,000 \times 20\% = 200$ BRL interest is credited to the account. The account then earns $1,200 \times 20\% = 240$ BRL in the second year
- A rate of 1% per month is equivalent to a simple annual interest rate (nominal rate) of 12%, but allowing for the effect of compounding, the annual equivalent compound rate is 12.68% per annum $(1.01^{12} - 1)$.
- The interest on corporate bonds and government bonds is usually payable twice yearly. The amount of interest paid (each six months) is the disclosed interest rate divided by two and multiplied by the principal. The yearly compounded rate is higher than the disclosed rate.
- Canadian mortgage loans are generally compounded semi-annually with monthly (or more frequent) payments.^[2]
- U.S. mortgages use an amortizing loan, not compound interest. With these loans, an amortization schedule is used to determine how to apply payments toward principal and interest. Interest generated on these loans is not added to the principal, but rather is paid off monthly as the payments are applied.
- It is sometimes mathematically simpler, e.g. in the valuation of derivatives, to use *continuous compounding* which is the limit as the compounding period approaches zero. Continuous compounding in pricing these instruments is a natural consequence of Itô calculus, where financial derivatives are valued at ever increasing frequency until the limit is approached and the derivative is valued in continuous time.

Discount instruments

- US and Canadian TBills (short term Government debt) have a different convention. Their interest is calculated on a discount basis as $(100 - P)/P \times 360/n$, where P is the price paid. Instead of normalizing it to a year the interest is prorated by the number of days: $(365/t) \times 100$. (See day count convention).

Mathematics of interest rate on loans

Periodic compounding

The total accumulated value, including the principal sum P plus compounded interest I , is given by the formula:

$$P' = P \left(1 + \frac{r}{n} \right)^{nt}$$

where:

P is the original principal sum

P' is the new principal sum

r is the nominal annual interest rate

n is the compounding frequency

t is the overall length of time the interest is applied (usually expressed in years).

The total compound interest generated is the final value minus the initial principal:

$$I = P \left(1 + \frac{r}{n} \right)^{nt} - P$$

Example 1

Suppose a principal amount of \$1,500 is deposited in a bank paying an annual interest rate of 4.3%, compounded quarterly. Then the balance after 6 years is found by using the formula above, with $P = 1500$, $r = 0.043$ (4.3%), $n = 4$, and $t = 6$:

$$P' = 1,500 \times \left(1 + \frac{0.043}{4} \right)^{4 \times 6} \approx 1,938.84$$

So the new principal P' after 6 years is approximately \$1,938.84.

Subtracting the original principal from this amount gives the amount of interest received:

$$1,938.84 - 1,500 = 438.84$$

Example 2

Suppose the same amount \$1,500 is compounded biennially (every 2 years).

Then the balance after 6 years is found by using the formula above, with $P = 1500$, $r = 0.043$ (4.3%), $n = 1/2 = 0.5$ (the interest is compounded every two years), and $t = 6$:

$$P' = 1,500 \times \left(1 + \frac{0.043}{0.5} \right)^{0.5 \times 6} \approx 1,921.24$$

So, the balance after 6 years is approximately \$1,921.24.

The amount of interest received can be calculated by subtracting the principal from this amount.

$$1,921.24 - 1,500 = 421.24$$

The interest is less compared with the previous case, as a result of the lower compounding frequency

Accumulation function

Since the principal P is simply a coefficient, it is often dropped for simplicity, and the resulting accumulation function is used instead. The accumulation function shows what \$1 grows to after any length of time. Accumulation functions for simple and compound interest are

$$a(t) = 1 + tr$$

$$a(t) = \left(1 + \frac{r}{n}\right)^{nt}$$

Continuous compounding

As n , the number of compounding periods per year, increases without limit, we have the case known as continuous compounding, in which case the effective annual rate approaches an upper limit of $e^r - 1$, where e is a mathematical constant that is the base of the natural logarithm

Continuous compounding can be thought of as making the compounding period infinitesimally small, achieved by taking the limit as n goes to infinity. See definitions of the exponential function for the mathematical proof of this limit. The amount after t periods of continuous compounding can be expressed in terms of the initial amount P_0 as

$$P(t) = P_0 e^{rt}.$$

Force of interest

As the number of compounding periods n reaches infinity in continuous compounding, the continuous compound interest is referred to as the force of interest δ .

In mathematics, the accumulation functions are often expressed in terms of e , the base of the natural logarithm. This facilitates the use of calculus to manipulate interest formulae.

For any continuously differentiable accumulation function $a(t)$, the force of interest, or more generally the logarithmic or continuously compounded return is a function of time defined as follows:

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{d}{dt} \ln a(t)$$

This is the logarithmic derivative of the accumulation function.

Conversely:

$$a(t) = e^{\int_0^t \delta_s ds}, \text{ (since } a(0) = 1; \text{ this can be viewed as a particular case of a } \underline{\text{product integral}}).$$

When the above formula is written in differential equation format, then the force of interest is simply the coefficient of amount of change:

$$da(t) = \delta_t a(t) dt$$

For compound interest with a constant annual interest rate r , the force of interest is a constant, and the accumulation function of compounding interest in terms of force of interest is a simple power of e :

$$\delta = \ln(1 + r) \text{ or}$$

$$a(t) = e^{t\delta}$$

The force of interest is less than the annual effective interest rate, but more than the annual effective discount rate. It is the reciprocal of the e-folding time. See also notation of interest rates

A way of modeling the force of inflation is with Stoodley's formula $\delta_t = p + \frac{s}{1 + rse^{st}}$ where p, r and s are estimated.

Compounding basis

To convert an interest rate from one compounding basis to another compounding basis, use

$$r_2 = \left[\left(1 + \frac{r_1}{n_1} \right)^{\frac{n_1}{n_2}} - 1 \right] n_2,$$

where r_1 is the interest rate with compounding frequency n_1 , and r_2 is the interest rate with compounding frequency n_2 .

When interest is continuously compounded use

$$\delta = n \ln \left(1 + \frac{r}{n} \right),$$

where δ is the interest rate on a continuous compounding basis, and r is the stated interest rate with a compounding frequency n .

Monthly amortized loan or mortgage payments

The interest on loans and mortgages that are amortized—that is, have a smooth monthly payment until the loan has been paid off—is often compounded monthly. The formula for payments is found from the following argument.

Exact formula for monthly payment

An exact formula for the monthly payment c is

$$c = \frac{Pr}{1 - \frac{1}{(1+r)^n}}$$

or equivalently

$$c = \frac{Pr}{1 - e^{-n \ln(1+r)}}$$

where:

c = monthly payment
 P = principal
 r = monthly interest rate
 n = number of payment periods

This can be derived by considering how much is left to be repaid after each month.

The Principal remaining after the first month is

$$P_1 = (1 + r)P - c$$

i.e. the initial amount has increased less the payment.

If the whole loan is repaid after one month then

$$P_1 = 0, \text{ so } P = \frac{c}{1+r}$$

After the second month $P_2 = (1+r)P_1 - c$ is left, so

$$P_2 = (1+r)((1+r)P - c) - c$$

If the whole loan was repaid after two months,

$$P_2 = 0, \text{ so } P = \frac{c}{1+r} + \frac{c}{(1+r)^2}$$

This equation generalises for a term of n months $P = c \sum_{j=1}^n \frac{1}{(1+r)^j}$. This is a geometric series which has the sum

$$P = \frac{c}{r} \left(1 - \frac{1}{(1+r)^n} \right)$$

which can be rearranged to give

$$c = \frac{Pr}{1 - \frac{1}{(1+r)^n}} = \frac{Pr}{1 - e^{-n \ln(1+r)}}$$

Spreadsheet Formula

In spreadsheets, the **PMT()** function is used. The syntax is:

$$PMT(\text{interest_rate}, \text{number_payments}, \text{present_value}, \text{future_value}, [\text{Type}])$$

See [Excel](#), [Mac Numbers](#), [Libreoffice](#), [Open Office](#) for more details.

For example, for interest rate of 6% (0.06/12), 25 years * 12 p.a., PV of \$150,000, FV of 0, type of 0 gives:

$$\begin{aligned} &= PMT(0.06/12, 25 * 12, 150000, 0, 0) \\ &= \$966.45 \end{aligned}$$

Approximate formula for monthly payment

A formula that is accurate to within a few percent can be found by noting that for typical U.S. note rates ($I < 8\%$ and terms $T=10-30$ years), the monthly note rate is small compared to $1r \ll 1$ so that the $\ln(1+r) \approx r$ which yields a simplification so that

$$c \approx \frac{Pr}{1 - e^{-nr}} = \frac{P}{n} \frac{nr}{1 - e^{-nr}}$$

which suggests defining auxiliary variables

$$Y \equiv nr = IT$$

$$c_0 \equiv \frac{P}{n}.$$

Here c_0 is the monthly payment required for a zero-interest loan paid off in n installments. In terms of these variables the approximation can be written

$$c \approx c_0 \frac{Y}{1 - e^{-Y}},$$

The function $f(Y) \equiv \frac{Y}{1 - e^{-Y}} - \frac{Y}{2}$ is even:

$$f(Y) = f(-Y)$$

implying that it can be expanded in even powers of Y .

It follows immediately that $\frac{Y}{1 - e^{-Y}}$ can be expanded in even powers of Y plus the single term: $Y/2$.

It will prove convenient then to define

$$X = \frac{1}{2}Y = \frac{1}{2}IT$$

so that

$$c \approx c_0 \frac{2X}{1 - e^{-2X}}$$

which can be expanded:

$$c \approx c_0 \left(1 + X + \frac{X^2}{3} - \frac{1}{45}X^4 + \dots \right)$$

where the ellipses indicate terms that are higher order in even powers of X . The expansion

$$P \approx P_0 \left(1 + X + \frac{X^2}{3} \right)$$

is valid to better than 1% provided $X \leq 1$.

Example of mortgage payment

For a \$10,000 mortgage with a term of 30 years and a note rate of 4.5%, payable yearly, we find:

$$T = 30$$

$$I = 0.045$$

which gives

$$X = \frac{1}{2}IT = .675$$

so that

$$P \approx P_0 \left(1 + X + \frac{1}{3}X^2 \right) = \$333.33(1 + .675 + .675^2/3) = \$608.96$$

The exact payment amount is $P = \$608.02$ so the approximation is an overestimate of about a sixth of a percent.

History

Compound interest was once regarded as the worst kind of usury and was severely condemned by Roman law and the common laws of many other countries.^[3]

The Florentine merchant Francesco Balducci Pegolotti provided a table of compound interest in his book *Pratica della mercatura* of about 1340. It gives the interest on 100 lire, for rates from 1% to 8%, for up to 20 years.^[4] The *Summa de arithmetica* of Luca Pacioli (1494) gives the Rule of 72, stating that to find the number of years for an investment at compound interest to double, one should divide the interest rate into 72.

Richard Witt's book *Arithmetical Questions* published in 1613, was a landmark in the history of compound interest. It was wholly devoted to the subject (previously called **anatocism**), whereas previous writers had usually treated compound interest briefly in just one chapter in a mathematical textbook. Witt's book gave tables based on 10% (the then maximum rate of interest allowable on loans) and on other rates for different purposes, such as the valuation of property leases. Witt was a London mathematical practitioner and his book is notable for its clarity of expression, depth of insight and accuracy of calculation, with 124 worked examples.^{[5][6]}

Jacob Bernoulli discovered the constant e in 1683 by studying a question about compound interest.

See also

- Credit card interest
- Exponential growth
- Fisher equation
- Interest
- Interest rate
- Rate of return
- Rate of return on investment
- Real versus nominal value (economics)
- Yield curve
- e (mathematical constant)

References

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2. http://laws.justice.gc.ca/en/showdoc/cs/l-15/bo-ga:s_6/en#anchorbo-ga:s_6 Interest Act (Canada), *Department of Justice*. The Interest Act specifies that interest is not recoverable unless the mortgage loan contains a statement showing the rate of interest chargeable, "calculated yearly or half-yearly, not in advance." In practice, banks use the half-yearly rate.
3. This article incorporates text from a publication now in the public domain: Chambers, Ephraim ed. (1728). '**article name needed**. *Cyclopædia, or an Universal Dictionary of Arts and Sciences* (first ed.). James and John Knapton, et al.
4. Evans, Allan (1936). *Francesco Balducci Pegolotti, La Pratica della Mercatura* Cambridge, Mass. pp. 301–2.
5. Lewin, C G (1970). "An Early Book on Compound Interest - Richard Witt's *Arithmetical Questions*" *Journal of the Institute of Actuaries* **96** (1): 121–132.
6. Lewin, C G (1981). "Compound Interest in the Seventeenth Century" *Journal of the Institute of Actuaries* **108** (3): 423–442.

Compound Interest Calculator

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