

Numeral system

A **numeral system** (or **system of numeration**) is a writing system for expressing numbers; that is, a mathematical notation for representing numbers of a given set, using digits or other symbols in a consistent manner. It can be seen as the context that allows the symbols "11" to be interpreted as the binary symbol for *three*, the decimal symbol for *eleven*, or a symbol for other numbers in different bases.

The number the numeral represents is called its value.

Ideally, a numeral system will:

- Represent a useful set of numbers (e.g. all integers, or rational numbers)
- Give every number represented a unique representation (or at least a standard representation)
- Reflect the algebraic and arithmetic structure of the numbers.

For example, the usual decimal representation of whole numbers gives every nonzero whole number a unique representation as a finite sequence of digits, beginning with a non-zero digit. However, when decimal representation is used for the rational or real numbers, such numbers in general have an infinite number of representations, for example 2.31 can also be written as 2.310, 2.310000, 2.30999999..., etc., all of which have the same meaning except for some scientific and other contexts where greater precision is implied by a larger number of figures shown.

Numeral systems are sometimes called number systems, but that name is ambiguous, as it could refer to different systems of numbers, such as the system of real numbers, the system of complex numbers, the system of *p*-adic numbers, etc. Such systems are, however, not the topic of this article.

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Main numeral systems

The most commonly used system of numerals is the Hindu–Arabic numeral system.^[1] Two Indian mathematicians are credited with developing it. Aryabhata of Kusumapura developed the place-value notation in the 5th century and a century later Brahmagupta introduced the symbol for zero. The numeral system and the zero concept, developed by the Hindus in India, slowly spread to other surrounding countries due to their commercial and military activities with India. The Arabs adopted and modified it. Even today, the Arabs call the numerals which they use "Rakam Al-Hind" or the Hindu numeral system. The Arabs translated Hindu texts on numerology and spread them to the western world due to their trade links with them. The Western world modified them and called them the Arabic numerals, as they learned them from the Arabs. Hence the current western numeral system is the modified version of the Hindu numeral system developed in India. It also exhibits a great similarity to the Sanskrit–Devanagari notation, which is still used in India and neighbouring Nepal.

The simplest numeral system is the unary numeral system, in which every natural number is represented by a corresponding number of symbols. If the symbol / is chosen, for example, then the number seven would be represented by // // // // // /. Tally marks represent one such system still in common use. The unary system is only useful for small numbers, although it plays an important role in theoretical computer science. Elias gamma coding, which is commonly used in data compression, expresses arbitrary-sized numbers by using unary to indicate the length of a binary numeral.

The unary notation can be abbreviated by introducing different symbols for certain new values. Very commonly, these values are powers of 10; so for instance, if / stands for one, – for ten and + for 100, then the number 304 can be compactly represented as +++ // // and the number 123 as + – – // // without any need for zero. This is called sign-value notation. The ancient Egyptian numeral system was of this type, and the Roman numeral system was a modification of this idea.

More useful still are systems which employ special abbreviations for repetitions of symbols; for example, using the first nine letters of the alphabet for these abbreviations, with A standing for "one occurrence", B "two occurrences", and so on, one could then write C+ D/ for the number 304. This system is used when writing Chinese numerals and other East Asian numerals based on Chinese. The number system of the English language is of this type ("three hundred [and] four"), as are those of other spoken languages, regardless of what written systems they have adopted. However, many languages use mixtures of bases, and other features, for instance 79 in French is *soixante dix-neuf* ($60 + 10 + 9$) and in Welsh is *pedwar ar bymtheg a thrigain* ($4 + (5 + 10) + (3 \times 20)$) or (somewhat archaic) *pedwar ugain namyn un* ($4 \times 20 - 1$). In English, one could say "four score less one", as in the famous Gettysburg Address representing "87 years ago" as "four score and seven years ago".

More elegant is a positional system, also known as place-value notation. Again working in base 10, ten different digits 0, ..., 9 are used and the position of a digit is used to signify the power of ten that the digit is to be multiplied with, as in $304 = 3 \times 100 + 0 \times 10 + 4 \times 1$ or more precisely $3 \times 10^2 + 0 \times 10^1 + 4 \times 10^0$. Note that zero, which is not needed in the other systems, is of crucial importance here, in order to be able to "skip" a power. The Hindu–Arabic numeral system, which originated in India and is now used throughout the world, is a positional base 10 system.

Arithmetic is much easier in positional systems than in the earlier additive ones; furthermore, additive systems need a large number of different symbols for the different powers of 10; a positional system needs only ten different symbols (assuming that it uses base 10).

The positional decimal system is presently universally used in human writing. The base 1000 is also used (albeit not universally), by grouping the digits and considering a sequence of three decimal digits as a single digit. This is the meaning of the common notation 1,000,234,567 used for very large numbers.

In computers, the main numeral systems are based on the positional system in base 2 (binary numeral system), with two binary digits, 0 and 1. Positional systems obtained by grouping binary digits by three (octal numeral system) or four (hexadecimal numeral system) are commonly used. For very large integers, bases 2^{32} or 2^{64} (grouping binary digits by 32 or 64, the length of the machine word) are used, as, for example, in GMP.

The numerals used when writing numbers with digits or symbols can be divided into two types that might be called the arithmetic numerals (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and the geometric numerals (1, 10, 100, 1000, 10000 ...), respectively. The sign-value systems use only the geometric numerals and the positional systems use only the arithmetic numerals. A sign-value system does not need arithmetic numerals because they are made by repetition (except for the Ionic system), and a positional system does not need geometric numerals because they are made by position. However, the spoken language uses *both* arithmetic and geometric numerals.

In certain areas of computer science, a modified base k positional system is used, called bijective numeration with digits 1, 2, ..., k ($k \geq 1$), and zero being represented by an empty string. This establishes a bijection between the set of all such digit-strings and the set of non-negative integers, avoiding the non-uniqueness caused by leading zeros. Bijective base- k numeration is also called k -adic notation, not to be confused with p -adic numbers. Bijective base 1 is the same as unary.

Positional systems in detail

In a positional base b numeral system (with b a natural number greater than 1 known as the radix), b basic symbols (or digits) corresponding to the first b natural numbers including zero are used. To generate the rest of the numerals, the position of the symbol in the figure is used. The symbol in the last position has its own value, and as it moves to the left its value is multiplied **by**

For example, in the decimal system (base 10), the numeral 4327 means $(4 \times 10^3) + (3 \times 10^2) + (2 \times 10^1) + (7 \times 10^0)$, noting that $10^0 = 1$.

In general, if b is the base, one writes a number in the numeral system of base b by expressing it in the form $a_n b^n + a_{n-1} b^{n-1} + a_{n-2} b^{n-2} + \dots + a_0 b^0$ and writing the enumerated digits $a_n a_{n-1} a_{n-2} \dots a_0$ in descending order. The digits are natural numbers between 0 and $b - 1$, inclusive.

If a text (such as this one) discusses multiple bases, and if ambiguity exists, the base (itself represented in base 10) is added in subscript to the right of the number, like this: number_{base}. Unless specified by context, numbers without subscript are considered to be decimal.

By using a dot to divide the digits into two groups, one can also write fractions in the positional system. For example, the base 2 numeral 10.11 denotes $1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2} = 2.75$.

In general, numbers in the base b system are of the form:

$$(a_n a_{n-1} \dots a_1 a_0 . c_1 c_2 c_3 \dots)_b = \sum_{k=0}^n a_k b^k + \sum_{k=1}^{\infty} c_k b^{-k}.$$

The numbers b^k and b^{-k} are the weights of the corresponding digits. The position k is the logarithm of the corresponding weight w , that is $k = \log_b w = \log_b b^k$. The highest used position is close to the order of magnitude of the number

The number of tally marks required in the unary numeral system for *describing the weight* would have been w . In the positional system, the number of digits required to describe it is only $k + 1 = \log_b w + 1$, for $k \geq 0$. For example, to describe the weight 1000 then four digits are needed because $\log_{10} 1000 + 1 = 3 + 1$. The number of digits required to *describe the position* is $\log_b k + 1 = \log_b \log_b w + 1$ (in positions 1, 10, 100,... only for simplicity in the decimal example).

Position	3	2	1	0	−1	−2	...
Weight	b^3	b^2	b^1	b^0	b^{-1}	b^{-2}	...
Digit	a_3	a_2	a_1	a_0	c_1	c_2	...
Decimal example weight	1000	100	10	1	0.1	0.01	...
Decimal example digit	4	3	2	7	0	0	...

Note that a number has a terminating or repeating expansion if and only if it is rational; this does not depend on the base. A number that terminates in one base may repeat in another (thus $0.3_{10} = 0.0100110011001\dots_2$). An irrational number stays aperiodic (with an infinite number of non-repeating digits) in all integral bases. Thus, for example in base 2, $\pi = 3.1415926\dots_{10}$ can be written as the aperiodic $11.00100100001111\dots_2$.

Putting overscores, $\bar{n}$, or dots, $\dot{n}$, above the common digits is a convention used to represent repeating rational expansions. Thus:

$$14/11 = 1.272727272727\dots = 1.\overline{27} \quad \text{or} \quad 321.3217878787878\dots = 321.321\overline{78}.$$

If $b = p$ is a prime number, one can define base- p numerals whose expansion to the left never stops; these are called the p -adic numbers.

Generalized variable-length integers

More general is using amixed radix notation (here written little-endian) like $a_0 a_1 a_2$ for $a_0 + a_1 b_1 + a_2 b_1 b_2$, etc.

This is used in [punycode](#), one aspect of which is the representation of a sequence of non-negative integers of arbitrary size in the form of a sequence without delimiters, of "digits" from a collection of 36: a–z and 0–9, representing 0–25 and 26–35 respectively. A digit lower than a threshold value marks that it is the most-significant digit, hence the end of the number. The threshold value depends on the position in the number. For example, if the threshold value for the first digit is b (i.e. 1) then a (i.e. 0) marks the end of the number (it has just one digit), so in numbers of more than one digit, range is only b–9 (1–35), therefore the weight is 35 instead of 36. Suppose the threshold values for the second and third digits are c (2), then the third digit has a weight $34 \times 35 = 1190$ and we have the following sequence:

a (0), ba (1), ca (2), ..., 9a (35), bb (36), cb (37), ..., 9b (70), bca (71), ..., 99a (1260), bcb (1261), etc.

Unlike a regular based numeral system, there are numbers like 9b where 9 and b each represents 35; yet the representation is unique because ac and aca are not allowed – the a would terminate the number

The flexibility in choosing threshold values allows optimization depending on the frequency of occurrence of numbers of various sizes.

The case with all threshold values equal to 1 corresponds to [bijective numeration](#), where the zeros correspond to separators of numbers with digits which are non-zero.

See also

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- [Numerical Mechanisms and Children's Concept of Numbers](#)
 - [Online conversion of fractional numbers between numeral systems](#)
 - [Ethiopic Numeral Names, Abera Molla](#)
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