

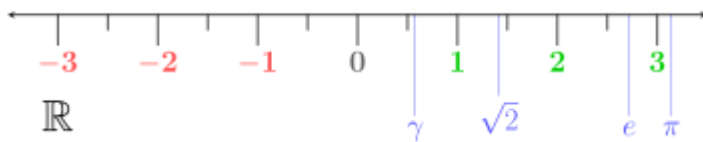
Real number

In mathematics, a **real number** is a value that represents a quantity along a line. The adjective *real* in this context was introduced in the 17th century by René Descartes who distinguished between real and imaginary roots of polynomials.

The real numbers include all the rational numbers, such as the integer -5 and the fraction $4/3$, and all the irrational numbers, such as $\sqrt{2}$ (1.41421356..., the square root of 2, an irrational algebraic number). Included within the irrationals are the transcendental numbers such as π (3.14159265...). Real numbers can be thought of as points on an infinitely long line called the number line or real line, where the points corresponding to integers are equally spaced. Any real number can be determined by a possibly infinite decimal representation, such as that of 8.632, where each consecutive digit is measured in units one tenth the size of the previous one. The real line can be thought of as a part of the complex plane, and complex numbers include real numbers.



A symbol of the set of **real numbers** ($\mathbb{R}$)



Real numbers can be thought of as points on an infinitely long number line

These descriptions of the real numbers are not sufficiently rigorous by the modern standards of pure mathematics. The discovery of a suitably rigorous definition of the real numbers – indeed, the realization that a better definition was needed – was one of the most important developments of 19th-century mathematics. The current standard axiomatic definition is that real numbers form the unique Dedekind-complete ordered field ($\mathbb{R}$; + ; · ; <), up to an isomorphism,^[a] whereas popular constructive definitions of real numbers include declaring them as equivalence classes of Cauchy sequences of rational numbers, Dedekind cuts, or infinite decimal representations, together with precise interpretations for the arithmetic operations and the order relation. All these definitions satisfy the axiomatic definition and are thus equivalent.

The reals are uncountable; that is: while both the set of all natural numbers and the set of all real numbers are infinite sets, there can be no one-to-one function from the real numbers to the natural numbers: the cardinality of the set of all real numbers (denoted $\mathfrak{c}$ and called cardinality of the continuum) is strictly greater than the cardinality of the set of all natural numbers (denoted $\aleph_0$ 'aleph-naught'). The statement that there is no subset of the reals with cardinality strictly greater than $\aleph_0$ and strictly smaller than $\mathfrak{c}$ is known as the continuum hypothesis (CH). It is known to be neither provable nor refutable using the axioms of Zermelo–Fraenkel set theory including the axiom of choice (ZFC), the standard foundation of modern mathematics, in the sense that some models of ZFC satisfy CH, while others violate it.

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History

Simple fractions were used by the Egyptians around 1000 BC; the Vedic "Sulba Sutras" ("The rules of chords") in, c. 600 BC, include what may be the first "use" of irrational numbers. The concept of irrationality was implicitly accepted by early Indian mathematicians since Manava (c. 750–690 BC), who were aware that the square roots of certain numbers such as 2 and 61 could not be exactly determined.^[1] Around 500 BC, the Greek mathematicians led by Pythagoras realized the need for irrational numbers, in particular the irrationality of the square root of 2.

The Middle Ages brought the acceptance of zero, negative, integral, and fractional numbers, first by Indian and Chinese mathematicians and then by Arabic mathematicians, who were also the first to treat irrational numbers as algebraic objects,^[2] which was made possible by the development of algebra. Arabic mathematicians merged the concepts of "number" and "magnitude" into a more general idea of real numbers.^[3] The Egyptian mathematician Abū Kāmil Shujā ibn Aslam (c. 850–930) was the first to accept irrational numbers as solutions to quadratic equations or as coefficients in an equation, often in the form of square roots, cube roots and fourth roots.^[4]

In the 16th century, Simon Stevin created the basis for modern decimal notation, and insisted that there is no difference between rational and irrational numbers in this regard.

In the 17th century, Descartes introduced the term "real" to describe roots of a polynomial, distinguishing them from "imaginary" ones.

In the 18th and 19th centuries, there was much work on irrational and transcendental numbers. Johann Heinrich Lambert (1761) gave the first flawed proof that π cannot be rational; Adrien-Marie Legendre (1794) completed the proof,^[5] and showed that π is not the square root of a rational number.^[6] Paolo Ruffini (1799) and Niels Henrik Abel (1842) both constructed proofs of the Abel–Ruffini theorem: that the general quintic or higher equations cannot be solved by a general formula involving only arithmetical operations and roots.

Évariste Galois (1832) developed techniques for determining whether a given equation could be solved by radicals, which gave rise to the field of Galois theory. Joseph Liouville (1840) showed that neither e nor e^2 can be a root of an integer quadratic equation, and then established the existence of transcendental numbers; Georg Cantor (1873) extended and greatly simplified this proof.^[7] Charles



Real numbers ($\mathbb{R}$) include the rational numbers ($\mathbb{Q}$), which include the integers ($\mathbb{Z}$), which include the natural numbers ($\mathbb{N}$)

Hermite (1873) first proved that e is transcendental, and Ferdinand von Lindemann (1882), showed that π is transcendental. Lindemann's proof was much simplified by Weierstrass (1885), still further by David Hilbert (1893), and has finally been made elementary by Adolf Hurwitz and Paul Gordan.^[8]

The development of calculus in the 18th century used the entire set of real numbers without having defined them cleanly. The first rigorous definition was given by Georg Cantor in 1871. In 1874, he showed that the set of all real numbers is uncountably infinite but the set of all algebraic numbers is countably infinite. Contrary to widely held beliefs, his first method was not his famous diagonal argument, which he published in 1891. See Cantor's first uncountability proof

Definition

The real number system $(\mathbb{R}; +; \cdot; <)$ can be defined axiomatically up to an isomorphism, which is described hereafter. There are also many ways to construct "the" real number system, for example, starting from natural numbers, then defining rational numbers algebraically, and finally defining real numbers as equivalence classes of their Cauchy sequences or as Dedekind cuts, which are certain subsets of rational numbers. Another possibility is to start from some rigorous axiomatization of Euclidean geometry (Hilbert, Tarski, etc.) and then define the real number system geometrically. From the structuralist point of view all these constructions are on equal footing.

Axiomatic approach

Let $\mathbb{R}$ denote the set of all real numbers. Then:

- The set $\mathbb{R}$ is a field, meaning that addition and multiplication are defined and have the usual properties.
- The field $\mathbb{R}$ is ordered, meaning that there is a total order $\geq$ such that, for all real numbers x, y and z :
 - if $x \geq y$ then $x + z \geq y + z$;
 - if $x \geq 0$ and $y \geq 0$ then $xy \geq 0$.
- The order is Dedekind-complete, that is: every non-empty subset S of $\mathbb{R}$ with an upper bound in $\mathbb{R}$ has a least upper bound (also called supremum) in $\mathbb{R}$.

The last property is what differentiates the reals from the rationals. For example, the set of rationals with square less than 2 has a rational upper bound (e.g., 1.5) but no rational least upper bound, because the square root of 2 is not rational.

These properties imply Archimedean property (which is not implied by other definitions of completeness). That is, the set of integers is not upper-bounded in the reals. In fact, if this were false, then the integers would have a least upper bound N ; then, $N - 1$ would not be an upper bound, and there would be an integer n such that $n > N - 1$, and thus $n + 1 > N$, which is a contradiction with the upper-bound property of N .

The real numbers are uniquely specified by the above properties. More precisely, given any two Dedekind-complete ordered fields $\mathbb{R}_1$ and $\mathbb{R}_2$, there exists a unique field isomorphism from $\mathbb{R}_1$ to $\mathbb{R}_2$, allowing us to think of them as essentially the same mathematical object.

For another axiomatization of $\mathbb{R}$, see Tarski's axiomatization of the reals

Construction from the rational numbers

The real numbers can be constructed as a completion of the rational numbers in such a way that a sequence defined by a decimal or binary expansion like (3; 3.1; 3.14; 3.141; 3.1415; ...) converges to a unique real number, in this case π . For details and other constructions of real numbers, see construction of the real numbers

Properties

Basic properties

A real number may be either rational or irrational; either algebraic or transcendental; and either positive, negative, or zero. Real numbers are used to measure continuous quantities. They may be expressed by decimal representations that have an infinite sequence of digits to the right of the decimal point; these are often represented in the same form as 324.823122147... The ellipsis (three dots) indicates that there would still be more digits to come.

More formally, the real numbers have the two basic properties of being an ordered field, and having the least upper bound property. The first says that real numbers comprise a field, with addition and multiplication as well as division by non-zero numbers, which can be totally ordered on a number line in a way compatible with addition and multiplication. The second says that, if a non-empty set of real numbers has an upper bound, then it has a real least upper bound. The second condition distinguishes the real numbers from the rational numbers: for example, the set of rational numbers whose square is less than 2 is a set with an upper bound (e.g. 1.5) but no (rational) least upper bound: hence the rational numbers do not satisfy the least upper bound property

Completeness

A main reason for using real numbers is that the reals contain all limits. More precisely, a sequence of real numbers has a limit, which is a real number, if (and only if) its elements eventually come and remain arbitrarily close to each other. This is formally defined in the following, and means that the reals are complete (in the sense of metric spaces or uniform spaces, which is a different sense than the Dedekind completeness of the order in the previous section). :

A sequence (x_n) of real numbers is called a Cauchy sequence if for any $\epsilon > 0$ there exists an integer N (possibly depending on ϵ) such that the distance $|x_n - x_m|$ is less than ϵ for all n and m that are both greater than N . This definition, originally provided by Cauchy, formalizes the fact that the x_n eventually come and remain arbitrarily close to each other

A sequence (x_n) *converges to the limit* x if its elements eventually come and remain arbitrarily close to x , that is, if for any $\epsilon > 0$ there exists an integer N (possibly depending on ϵ) such that the distance $|x_n - x|$ is less than ϵ for n greater than N .

Every convergent sequence is a Cauchy sequence, and the converse is true for real numbers, and this means that the topological space of the real numbers is complete.

The set of rational numbers is not complete. For example, the sequence (1; 1.4; 1.41; 1.414; 1.4142; 1.41421; ...), where each term adds a digit of the decimal expansion of the positive square root of 2, is Cauchy but it does not converge to a rational number (in the real numbers, in contrast, it converges to the positive square root of 2).

The completeness property of the reals is the basis on which calculus, and, more generally mathematical analysis are built. In particular, the test that a sequence is a Cauchy sequence allows proving that a sequence has a limit, without computing it, and even without knowing it.

For example, the standard series of the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges to a real number for every x , because the sums

$$\sum_{n=N}^M \frac{x^n}{n!}$$

can be made arbitrarily small (independently of M) by choosing N sufficiently large. This proves that the sequence is Cauchy, and thus converges, showing that e^x is well defined for every x .

"The complete ordered field"

The real numbers are often described as "the complete ordered field", a phrase that can be interpreted in several ways.

First, an order can be lattice-complete. It is easy to see that no ordered field can be lattice-complete, because it can have no largest element (given any element z , $z + 1$ is larger), so this is not the sense that is meant.

Additionally, an order can be Dedekind-complete, as defined in the section **Axioms**. The uniqueness result at the end of that section justifies using the word "the" in the phrase "complete ordered field" when this is the sense of "complete" that is meant. This sense of completeness is most closely related to the construction of the reals from Dedekind cuts, since that construction starts from an ordered field (the rationals) and then forms the Dedekind-completion of it in a standard way.

These two notions of completeness ignore the field structure. However, an ordered group (in this case, the additive group of the field) defines a uniform structure, and uniform structures have a notion of completeness (topology) the description in the previous section. **Completeness** is a special case. (We refer to the notion of completeness in uniform spaces rather than the related and better known notion for metric spaces, since the definition of metric space relies on already having a characterization of the real numbers.) It is not true that $\mathbf{R}$ is the *only* uniformly complete ordered field, but it is the only uniformly complete Archimedean field, and indeed one often hears the phrase "complete Archimedean field" instead of "complete ordered field". Every uniformly complete Archimedean field must also be Dedekind-complete (and vice versa), justifying using "the" in the phrase "the complete Archimedean field". This sense of completeness is most closely related to the construction of the reals from Cauchy sequences (the construction carried out in full in this article), since it starts with an Archimedean field (the rationals) and forms the uniform completion of it in a standard way.

But the original use of the phrase "complete Archimedean field" was by David Hilbert, who meant still something else by it. He meant that the real numbers form the *largest* Archimedean field in the sense that every other Archimedean field is a subfield of $\mathbf{R}$. Thus $\mathbf{R}$ is "complete" in the sense that nothing further can be added to it without making it no longer an Archimedean field. This sense of completeness is most closely related to the construction of the reals from surreal numbers, since that construction starts with a proper class that contains every ordered field (the surreals) and then selects from it the *largest* Archimedean subfield.

Advanced properties

The reals are uncountable; that is: there are strictly more real numbers than natural numbers, even though both sets are infinite. In fact, the cardinality of the reals equals that of the set of subsets (i.e. the power set) of the natural numbers, and Cantor's diagonal argument states that the latter set's cardinality is strictly greater than the cardinality of $\mathbf{N}$. Since the set of algebraic numbers is countable, almost all real numbers are transcendental. The non-existence of a subset of the reals with cardinality strictly between that of the integers and the reals is known as the continuum hypothesis. The continuum hypothesis can neither be proved nor be disproved; it is independent from the axioms of set theory.

As a topological space, the real numbers are separable. This is because the set of rationals, which is countable, is dense in the real numbers. The irrational numbers are also dense in the real numbers, however they are uncountable and have the same cardinality as the reals.

The real numbers form a metric space: the distance between x and y is defined as the absolute value $|x - y|$. By virtue of being a totally ordered set, they also carry an order topology; the topology arising from the metric and the one arising from the order are identical, but yield different presentations for the topology – in the order topology as ordered intervals, in the metric topology as epsilon-balls. The Dedekind cuts construction uses the order topology presentation, while the Cauchy sequences construction uses the metric topology presentation. The reals are a contractible (hence connected and simply connected), separable and complete metric space of Hausdorff dimension 1. The real numbers are locally compact but not compact. There are various properties that uniquely specify them; for instance, all unbounded, connected, and separable order topologies are necessarily homeomorphic to the reals.

Every nonnegative real number has a square root in $\mathbf{R}$, although no negative number does. This shows that the order on $\mathbf{R}$ is determined by its algebraic structure. Also, every polynomial of odd degree admits at least one real root: these two properties make $\mathbf{R}$ the premier example of a real closed field. Proving this is the first half of one proof of the fundamental theorem of algebra.

The reals carry a canonical measure, the Lebesgue measure, which is the Haar measure on their structure as a topological group normalized such that the unit interval $[0;1]$ has measure 1. There exist sets of real numbers that are not Lebesgue measurable, e.g. Vitali sets.

The supremum axiom of the reals refers to subsets of the reals and is therefore a second-order logical statement. It is not possible to characterize the reals with first-order logic alone: the Löwenheim–Skolem theorem implies that there exists a countable dense subset of the real numbers satisfying exactly the same sentences in first-order logic as the real numbers themselves. The set of hyperreal numbers satisfies the same first order sentences as $\mathbf{R}$. Ordered fields that satisfy the same first-order sentences as $\mathbf{R}$ are called nonstandard models of $\mathbf{R}$. This is what makes nonstandard analysis work; by proving a first-order statement in some nonstandard model (which may be easier than proving it in $\mathbf{R}$), we know that the same statement must also be true in $\mathbf{R}$.

The field $\mathbf{R}$ of real numbers is an extension field of the field $\mathbf{Q}$ of rational numbers, and $\mathbf{R}$ can therefore be seen as a vector space over $\mathbf{Q}$. Zermelo–Fraenkel set theory with the axiom of choice guarantees the existence of a basis of this vector space: there exists a set B of real numbers such that every real number can be written uniquely as a finite linear combination of elements of this set, using rational coefficients only, and such that no element of B is a rational linear combination of the others. However, this existence theorem is purely theoretical, as such a base has never been explicitly described.

The well-ordering theorem implies that the real numbers can be well-ordered if the axiom of choice is assumed: there exists a total order on $\mathbf{R}$ with the property that every non-empty subset of $\mathbf{R}$ has a least element in this ordering. (The standard ordering $\leq$ of the real numbers is not a well-ordering since e.g. an open interval does not contain a least element in this ordering.) Again, the existence of such a well-ordering is purely theoretical, as it has not been explicitly described. If $\mathbf{V}=\mathbf{L}$ is assumed in addition to the axioms of ZF, a well ordering of the real numbers can be shown to be explicitly definable by a formula^[9]

Applications and connections to other areas

Real numbers and logic

The real numbers are most often formalized using the Zermelo–Fraenkel axiomatization of set theory, but some mathematicians study the real numbers with other logical foundations of mathematics. In particular, the real numbers are also studied in reverse mathematics and in constructive mathematics^[10]

The hyperreal numbers as developed by Edwin Hewitt, Abraham Robinson and others extend the set of the real numbers by introducing infinitesimal and infinite numbers, allowing for building infinitesimal calculus in a way closer to the original intuitions of Leibniz, Euler, Cauchy and others.

Edward Nelson's internal set theory enriches the Zermelo–Fraenkel set theory syntactically by introducing a unary predicate "standard". In this approach, infinitesimals are (non-"standard") elements of the set of the real numbers (rather than being elements of an extension thereof, as in Robinson's theory).

The continuum hypothesis posits that the cardinality of the set of the real numbers is $\aleph_1$; i.e. the smallest infinite cardinal number after $\aleph_0$, the cardinality of the integers. Paul Cohen proved in 1963 that it is an axiom independent of the other axioms of set theory; that is: one may choose either the continuum hypothesis or its negation as an axiom of set theory without contradiction.

In physics

In the physical sciences, most physical constants such as the universal gravitational constant, and physical variables, such as position, mass, speed, and electric charge, are modeled using real numbers. In fact, the fundamental physical theories such as classical mechanics, electromagnetism, quantum mechanics, general relativity and the standard model are described using mathematical structures, typically smooth manifolds or Hilbert spaces, that are based on the real numbers, although actual measurements of physical quantities are of finite accuracy and precision

Physicists have occasionally suggested that a more fundamental theory would replace the real numbers with quantities that do not form a continuum, but such proposals remain speculative.^[11]

In computation

With some exceptions, most calculators do not operate on real numbers. Instead, they work with finite-precision approximations called floating-point numbers. In fact, most scientific computation uses floating-point arithmetic. Real numbers satisfy the usual rules of arithmetic, but floating-point numbers do not

Computers cannot directly store arbitrary real numbers with infinitely many digits. The achievable precision is limited by the number of bits allocated to store a number, whether as floating-point numbers or arbitrary-precision numbers. However, computer algebra systems can operate on irrational quantities exactly by manipulating formulas for them (such as $\sqrt{2}$, $\arcsin\left(\frac{2}{23}\right)$, or $\int_0^1 x^x dx$) rather than their rational or decimal approximation.^[12] It is not in general possible to determine whether two such expressions are equal (the constant problem).

A real number is called computable if there exists an algorithm that yields its digits. Because there are only countably many algorithms,^[13] but an uncountable number of reals, almost all real numbers fail to be computable. Moreover, the equality of two computable numbers is an undecidable problem. Some constructivists accept the existence of only those reals that are computable. The set of definable numbers is broader, but still only countable.

"Reals" in set theory

In set theory, specifically descriptive set theory, the Baire space is used as a surrogate for the real numbers since the latter have some topological properties (connectedness) that are a technical inconvenience. Elements of Baire space are referred to as "reals".

Vocabulary and notation

Mathematicians use the symbol **R**, or, alternatively, $\mathbb{R}$, the letter "R" in blackboard bold (encoded in Unicode as U+211D $\mathbb{R}$ DOUBLE-STRUCK CAPITAL R (HTML `ℝ` ;)), to represent the set of all real numbers. As this set is naturally endowed with the structure of a field, the expression *field of real numbers* is frequently used when its algebraic properties are under consideration.

The sets of positive real numbers and negative real numbers are often noted $\mathbf{R}^+$ and $\mathbf{R}^-$,^[14] respectively; $\mathbf{R}_+$ and $\mathbf{R}_-$ are also used.^[15] The non-negative real numbers can be noted $\mathbf{R}_{\geq 0}$ but one often sees this set noted $\mathbf{R}^+ \cup \{0\}$.^[14] In French mathematics, the *positive real numbers* and *negative real numbers* commonly include zero, and these sets are noted respectively $\mathbb{R}_+$ and $\mathbb{R}_-$.^[15] In this understanding, the respective sets without zero are called strictly positive real numbers and strictly negative real numbers, and are noted $\mathbb{R}_+^*$ and $\mathbb{R}_-^*$.^[15]

The notation $\mathbf{R}^n$ refers to the cartesian product of n copies of **R**, which is an n -dimensional vector space over the field of the real numbers; this vector space may be identified to the n -dimensional space of Euclidean geometry as soon as a coordinate system has been chosen in the latter. For example, a value from $\mathbf{R}^3$ consists of three real numbers and specifies the coordinates of a point in 3-dimensional space.

In mathematics, *real* is used as an adjective, meaning that the underlying field is the field of the real numbers (or *the real field*). For example, *real matrix*, *real polynomial* and *real Lie algebra*. The word is also used as a noun, meaning a real number (as in "the set of all reals").

Generalizations and extensions

The real numbers can be generalized and extended in several different directions:

- The complex numbers contain solutions to all polynomial equations and hence are an algebraically closed field unlike the real numbers. However the complex numbers are not an ordered field.

- The affinely extended real number system adds two elements $+\infty$ and $-\infty$. It is a compact space. It is no longer a field, or even an additive group, but it still has a total order; moreover, it is a complete lattice.
- The real projective line adds only one value ∞ . It is also a compact space. Again, it is no longer a field, or even an additive group. However it allows division of a non-zero element by zero. It has cyclic order described by a separation relation.
- The long real line pastes together $\aleph_1^* + \aleph_1$ copies of the real line plus a single point (here $\aleph_1^*$ denotes the reversed ordering of $\aleph_1$) to create an ordered set that is "locally" identical to the real numbers, but somehow longer; for instance, there is an order-preserving embedding of $\aleph_1$ in the long real line but not in the real numbers. The long real line is the largest ordered set that is complete and locally Archimedean. As with the previous two examples, this set is no longer a field or additive group.
- Ordered fields extending the reals are the hyperreal numbers and the surreal numbers both of them contain infinitesimal and infinitely large numbers and are therefore non-Archimedean ordered fields.
- Self-adjoint operators on a Hilbert space (for example, self-adjoint square complex matrices) generalize the reals in many respects: they can be ordered (though not totally ordered), they are complete, all their eigenvalues are real and they form a real associative algebra. Positive-definite operators correspond to the positive reals and normal operators correspond to the complex numbers.

See also

- Continued fraction
- Real analysis

Notes

- More precisely, given two complete totally ordered fields, there is a *unique* isomorphism between them. This implies that the identity is the unique field automorphism of the reals that is compatible with the ordering.

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 - The real numbers: Pythagoras to Stevin
 - The real numbers: Stevin to Hilbert
 - The real numbers: Attempts to understand
 - What are the "real numbers," really?
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