

R.3 The Laws of Exponents; Radicals

Properties of Positive Integer Exponents

Suppose m and n are positive integers and a and b are real numbers, then

1. $b^m b^n = b^{m+n}$

2. $\frac{b^m}{b^n} = b^{m-n}$, where $m > n$ and $b \neq 0$

$\frac{b^m}{b^n} = 1$, where $m = n$

$\frac{b^m}{b^n} = \frac{1}{b^{n-m}}$, where $m < n$ and $b \neq 0$

3. $(b^m)^n = b^{mn}$

4. $(ab)^n = a^n b^n$

5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where $b \neq 0$

$$\begin{aligned} b^2 \cdot b^3 &= b \cdot b \cdot b \cdot b \cdot b = b^5 \\ \frac{b^5}{b^2} &= \frac{b \cdot b \cdot b \cdot b \cdot b}{b \cdot b} = b^3 \\ \frac{b^3}{b^3} &= 1 = b^0 \\ \frac{b^1}{b^3} &= \frac{b}{b \cdot b \cdot b} = \frac{1}{b^2} \\ \frac{b}{b^5} &= \frac{1}{b^4} \end{aligned}$$

Although these five properties are defined for positive integer exponents, they are actually true for all integer exponents, including exponents that are negative integers or zero. So, for example, in property 1, for $n = 0$, $b^m b^0 = b^{m+0} = b^m$. Therefore, b^0 must be 1.

Zero Exponent Rule

If b is a real number such that $b \neq 0$, then $b^0 = 1$.

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Similarly, if we let $n = -m$, then $b^m \cdot b^{-m} = b^{m+(-m)} = b^0 = 1$ if and only if b^{-m} is the reciprocal of b^m or $b^{-m} = \frac{1}{b^m}$. This is known as the reciprocal rule of exponents.

Reciprocal Rule of Exponents

(Negative Exponent Rule)

If b is a real number such that $b \neq 0$ and m is a positive integer, then $b^{-m} = \frac{1}{b^m}$.

$$\frac{x^3}{x^8} = x^{-5} = \frac{1}{x^5} \quad 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

The following laws of exponents hold true for all integer exponents.

Laws of Exponents

Suppose m and n are integers and a and b are real numbers, then

1. $b^m b^n = b^{m+n}$ Product rule of exponents
2. $\frac{b^m}{b^n} = b^{m-n}$, where $b \neq 0$ Quotient rule of exponents
3. $(b^m)^n = b^{mn}$ Power rule of exponents
4. $(ab)^n = a^n b^n$ Product to power rule of exponents
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where $b \neq 0$ Quotient to power rule of exponents
6. $b^0 = 1$ Zero exponent rule
7. $b^{-m} = \frac{1}{b^m}$, where $b \neq 0$ Reciprocal rule of exponents

Simplify each exponential expression below. Write your answers using positive exponents. Assume all variables represent positive real numbers.

a. $-3^{-4} \cdot 3^4$

$$-3^0 = -1$$

b. $5z^4 \cdot (-9z^{-5})$

$$\frac{-45z^{-1}}{z} = \frac{-45}{z}$$

c. $\left(\frac{a^{-3}b^4c^{-6}}{2a^3b^{-4}c}\right)^{-3}$

$$= \frac{2^3 a^9 a^{15} c^{18} c^3}{b^{12} b^{12} b^{12}} = \frac{8a^{24} c^{21}}{b^{36}}$$

1. $3^2 \cdot 3^3$

$$3^5$$

2. $k^4 \cdot k^5$

$$k^9$$

3. $x^5 \cdot x^9 \cdot x^{12}$

$$x^{26}$$

4. $(a^5b^4)(a^7b^2)$

5. $\frac{t^9}{t^2}$

$$t^7$$

6. $(r^5)^{-7}$

$$\frac{1}{r^{35}}$$

7. $(w^{-8})^{-6}$

$$w^{48}$$

8. $\frac{n^1}{n^{-7}}$

$$\frac{n^1 \cdot n^7}{n^8} = \frac{n^8}{n^8} = 1$$

25. $(-5x^2y^5)^2$

$$25x^4y^{10}$$

26. $\frac{24x^3y^5}{4x^8y^{-7}}$

$$\frac{3y^{12}}{4x^5}$$

27. $\frac{(a^{-3}b)^{-2}}{(a^3b^{-1})^2}$

$$\frac{a^6b^{-2}}{a^6b^{-2}} = 1$$

28. $(16a^{-3}bc^{-6})(2ab)^{-5}$

$$\frac{16a^{-3}bc^{-6} \cdot 2^5 a^5 b^5 c^5}{1} = \frac{16b}{a^3 c^2} = \frac{16b}{a^3 c^2}$$

29. $\left(\frac{q^3p^4w^5}{q^{-3}p^{-4}w^{-5}}\right)^{-2}$

$$\left(\frac{q^6p^8w^{10}}{q^{-12}p^{-16}w^{-20}}\right)^{-2} = \frac{1}{q^{12}p^{16}w^{20}}$$

30. $\frac{(3^{-1}x^{-1}y^{-1})^{-2}(3x^{-4}y^3)(9x^{-2}y^3)^0}{(3x^{-3}y^{-5})^2}$

$$\frac{3^2 x^2 y^2 \cdot 3x^{-4}y^3 \cdot 1}{3^4 x^{-6}y^{-10}} = \frac{3x^2y^2 \cdot y^3 \cdot x^6y^{10}}{x^4} = \frac{3x^8y^{15}}{x^4} = 3x^4y^{15}$$

Simplifying Radical Expressions

$$\sqrt{25} = 5$$

A number a is said to be an n th root of b if $a^n = b$.

For example, 5 is a square root of 25 because $5^2 = 25$. The number -5 is also a square root of 25 because $(-5)^2 = 25$. Similarly, 6 and -6 are both square roots of 36 because $6^2 = 36$ and $(-6)^2 = 36$.

It follows that every positive real number has two real n th roots when n is an even integer. However, if n is an odd integer, there is only one root.

$$\sqrt[3]{8} = 2$$

For example, the only real cube root of 8 is 2 because $2^3 = 8$. (Note that $(-2)^3 \neq 8$.)

$$\sqrt[5]{32} =$$

We use the symbol $\sqrt[n]{b}$ to mean the positive n th root of b , also called the principal n th root of a number b .

The symbol $\sqrt[n]{\quad}$ called a radical sign the integer n is called the index, and b is called the radicand. The complete expression $\sqrt[n]{b}$ is called a radical. (or root)

Definition Principal n th Root

For any integer $n \geq 2$, the principal n th root of a number b is defined as

$$\sqrt[n]{b} = a \text{ if and only if } a^n = b.$$

If n is even and $b \geq 0$, then $a \geq 0$.

If n is even and $b < 0$, then a is not a real number.

If n is odd and $b \geq 0$, then $a \geq 0$.

If n is odd and $b < 0$, then $a < 0$.

$$\sqrt[3]{27} = 3 \text{ because } 3^3 = 27$$

$$\sqrt[4]{16} = 2$$

$$\sqrt[3]{-8} = -2$$

$$(-2ab)^5 = -32a^5b^5$$

Simplify each radical expression.

a. $\sqrt{144}$
12

b. $\sqrt[3]{\frac{1}{27}} = \frac{\sqrt[3]{1}}{\sqrt[3]{27}} = \frac{1}{3}$

c. $\sqrt[5]{-32a^5b^5} = -2ab$

d. $\sqrt[6]{-64}$ Not real

31. $\sqrt{169}$
13

32. $\sqrt{\frac{1}{25}} = \frac{\sqrt{1}}{\sqrt{25}} = \frac{1}{5}$

33. $\sqrt[3]{216}$
6

Sometimes a radical expression may contain radicands which are not perfect powers. To handle these, we must use some properties of radicals.

Properties of Radicals

If a and b are real numbers and $n \geq 2$ and $m \geq 2$ are integers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are **real numbers**, then

1. $\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
3. $\sqrt[n]{a^n} = a$ if n is odd
4. $\sqrt[n]{a^n} = |a|$ if n is even

$$\sqrt[5]{a^5 b^5} = ab$$

$$\sqrt[4]{x^4} = |x|$$

Simplify each radical expression.

a.

$$\sqrt{108} = 6\sqrt{3}$$

b.

$$\sqrt[3]{-40x^5y^3}$$

c.

$$\sqrt[4]{16x^4y^5}$$

$$\begin{aligned}\sqrt{50} &= \sqrt{25 \cdot 2} \\ &= \sqrt{25} \cdot \sqrt{2} \\ &= 5\sqrt{2}\end{aligned}$$

$$\frac{\sqrt{4 \cdot 27}}{\sqrt{4 \cdot 9 \cdot 3}} = \frac{\sqrt{4} \cdot \sqrt{9} \cdot \sqrt{3}}{2 \cdot 3 \cdot \sqrt{3}}$$

$$\begin{aligned}\sqrt[3]{-8 \cdot 5 \cdot x^3 \cdot x^2 \cdot y^3} \\ -2xy \sqrt[3]{5x^2}\end{aligned}$$

$$\begin{aligned}\sqrt[4]{16x^4y^4 \cdot y} \\ 2xy \sqrt[4]{y}\end{aligned}$$

37. $\sqrt{160}$

$$\begin{aligned}\sqrt{16 \cdot 10} &= \sqrt{16} \cdot \sqrt{10} \\ 4\sqrt{10}\end{aligned}$$

38. $\sqrt{363x^2}$

$$\begin{aligned}\sqrt{121 \cdot 3 \cdot x^2} \\ 11x\sqrt{3}\end{aligned}$$

39. $\sqrt[3]{250}$

$$\begin{aligned}\sqrt[3]{125 \cdot 2} \\ 5\sqrt[3]{2}\end{aligned}$$

40. $\sqrt[3]{-27x^4}$

$$\begin{aligned}\sqrt[3]{-27 \cdot x^3 \cdot x} \\ -3x \sqrt[3]{x}\end{aligned}$$

41. $\sqrt[4]{32z^5}$

$$\begin{aligned}\sqrt[4]{16 \cdot 2 \cdot z^4 \cdot z} \\ 2z \sqrt[4]{2z}\end{aligned}$$

42. $\sqrt{180x^{12}y^{27}}$

$$\begin{aligned}\sqrt{36 \cdot 5 \cdot x^{12} \cdot y^{26} \cdot y} \\ 6x^6y^{13} \sqrt{5y}\end{aligned}$$

43. $\sqrt[5]{\frac{x^{24}y^{13}}{x^9y^8}}$

$$\begin{aligned}\sqrt[5]{x^{15}y^5} \\ = x^3y\end{aligned}$$

We can add or subtract radicals only if the index and the radicand of each expression are the same. Simply add the coefficients.

Ex $\sqrt{7x} + 3\sqrt{7x} + 5\sqrt{7x}$

$$\boxed{9\sqrt{7x}}$$

$$5\sqrt[3]{16x} + 3\sqrt[3]{2x} - \sqrt[3]{2x^4}$$

$$\begin{aligned} &5\sqrt[3]{8 \cdot 2x} + 3\sqrt[3]{2x} - \sqrt[3]{2x^3 \cdot x} \\ &10\sqrt[3]{2x} + 3\sqrt[3]{2x} - x\sqrt[3]{2x} \\ &(13-x)\sqrt[3]{2x} \end{aligned}$$

Simplify each radical expression. Assume all variables represent positive real numbers.

a. $\sqrt{192}$

b. $(3\sqrt{2} + \sqrt{5})(2\sqrt{2} - 4\sqrt{5})$

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$$\begin{aligned} &3\sqrt{2} \cdot 2\sqrt{2} - 3\sqrt{2} \cdot 4\sqrt{5} + \sqrt{5} \cdot 2\sqrt{2} - \\ &\quad 6\sqrt{4} - 12\sqrt{10} + 2\sqrt{10} - \sqrt{5} \cdot 4\sqrt{5} \\ &6 \cdot 2 - 10\sqrt{10} - 4 \cdot 5 = \boxed{-8 - 10\sqrt{10}} \end{aligned}$$

c. $\sqrt[5]{2x^4y^4} \sqrt[5]{48xy^2}$

$$\begin{aligned} &\sqrt[5]{96x^5y^6} \\ &\sqrt[5]{32 \cdot 3 \cdot x^5 \cdot y^5 \cdot y} = 2xy \sqrt[5]{3y} \end{aligned}$$

d. $8\sqrt{2x^2} - 3\sqrt{28x} - 2\sqrt{8x^2}$

44. $\sqrt{6}(\sqrt{5} - \sqrt{11})$

$$\boxed{\sqrt{30} - \sqrt{66}}$$

45. $(3\sqrt{2} + 3\sqrt{7})(\sqrt{2} - 2\sqrt{7})$

$$\begin{aligned} &3\sqrt{4} - 6\sqrt{14} + 3\sqrt{14} - 6\sqrt{49} \\ &3 \cdot 2 - 3\sqrt{14} - 6 \cdot 7 = 6 - 3\sqrt{14} - 42 \\ &= \boxed{-36 - 3\sqrt{14}} \end{aligned}$$

46. $(4 + \sqrt{3})^2$

$$\begin{aligned} &(4 + \sqrt{3})(4 + \sqrt{3}) \\ &16 + 4\sqrt{3} + 4\sqrt{3} + \sqrt[3]{9} \\ &\boxed{19 + 8\sqrt{3}} \end{aligned}$$

47. $\sqrt[3]{x^2}(8\sqrt[3]{4x} - 5\sqrt[3]{x^5})$

$$8\sqrt[3]{4x^3} - 5\sqrt[3]{x^7}$$

$$\begin{aligned} &8\sqrt[3]{4x^3} - 5\sqrt[3]{x^6 \cdot x} \\ &\boxed{8x\sqrt[3]{4} - 5x^2\sqrt[3]{x}} \end{aligned}$$

48. $11\sqrt{5} + 3\sqrt{5}$

$$14\sqrt{5}$$

When a radical appears in the denominator of an algebraic expression, we typically multiply the numerator and denominator by an expression which will eliminate the radical in the denominator. This is called **rationalizing the denominator**.

Rationalize the denominator of each expression.

$$\begin{array}{ll} \text{a. } \sqrt{\frac{3}{2}} = \frac{\sqrt{3} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{6}}{2} & \text{b. } \frac{\sqrt[3]{375}}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{\sqrt[3]{750}}{\sqrt[3]{8}} = \frac{\sqrt[3]{125 \cdot 6}}{2} = \frac{5\sqrt[3]{6}}{2} \end{array}$$

$$\textcircled{51} \frac{\sqrt{5}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}}$$

$$= \frac{\sqrt{30}}{\sqrt{36}}$$

$$= \frac{\sqrt{30}}{6}$$

$$\textcircled{52} \sqrt{\frac{1}{7}}$$

$$\textcircled{53} \sqrt[3]{\frac{7}{2}}$$

$$\textcircled{54} \frac{\sqrt[3]{2y^5}}{\sqrt[3]{5x}}$$

$$\frac{\sqrt[3]{2y^5} \cdot \sqrt[3]{25x^2}}{\sqrt[3]{5x} \cdot \sqrt[3]{25x^2}}$$

$$= \frac{\sqrt[3]{50x^2y^5}}{\sqrt[3]{125x^3}}$$

$$= \frac{\sqrt[3]{50x^2y^3} \cdot y^2}{5x}$$

$$= \frac{y \sqrt[3]{50x^2y^2}}{5x}$$

We can use our knowledge of radicals to simplify expressions involving rational exponents. We start with the following definition.

Definition $b^{1/n}$

For any integer $n \geq 2$ and any real number b , we define the expression $b^{1/n}$ as

$$b^{1/n} = \sqrt[n]{b}$$

provided that the expression $\sqrt[n]{b}$ is a **real number***.

$$25^{1/2} = \sqrt{25} = 5$$

Simplify each expression.

a. $4^{1/2} = \sqrt{4} = 2$

b. $(-27)^{1/3} = \sqrt[3]{-27} = -3$

c. $\left(\frac{1}{64}\right)^{1/6} = \sqrt[6]{\frac{1}{64}} = \frac{\sqrt[6]{1}}{\sqrt[6]{64}} = \frac{1}{2}$

$$8^{1/3} = \sqrt[3]{8} = 2$$

58. $25^{1/2}$

59. $\left(\frac{1}{16}\right)^{1/4}$

60. $-81^{1/4}$

61. $16^{3/4}$

$$\sqrt[4]{\frac{1}{16}} = \frac{\sqrt[4]{1}}{\sqrt[4]{16}} = \frac{1}{2}$$

$$-\sqrt[4]{81} = -3$$

Notice that the exponent in #61 is not of the form $1/n$. To handle this, we need another definition:

$$b^{m/n} = \sqrt[n]{b^m}$$

Definition $b^{m/n}$

If $\frac{m}{n}$ is a **rational number** with $n \geq 2$ and b is a real number, then we define the expression $b^{m/n}$ as

$$b^{m/n} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$$

provided that the expression $\sqrt[n]{b}$ is a **real number***.

61. $16^{3/4}$

$$\begin{aligned} 16^{3/4} &= \sqrt[4]{16^3} = \sqrt[4]{4096} \\ &= \left(\sqrt[4]{16}\right)^3 = 2^3 \\ &= 8 \end{aligned}$$

a. $(9)^{3/2}$

$$\begin{aligned} 9^{3/2} &= \sqrt{9^3} = \sqrt{729} = 27 \\ &= \left(\sqrt{9}\right)^3 = 3^3 \\ &= 27 \end{aligned}$$

The laws of exponents that were stated earlier for integer exponents also hold true for rational exponents.

Simplify each expression.

$$\begin{aligned} \text{b. } (-32)^{-3/5} &= \frac{1}{(-32)^{3/5}} \\ &= \frac{1}{\sqrt[5]{(-32)^3}} = \frac{1}{(-2)^3} = \boxed{-\frac{1}{8}} \end{aligned}$$

$$\begin{aligned} \text{c. } \frac{(125x^4y^{-1/4})^{2/3}}{(x^2y)^{1/3}} &= \frac{125^{2/3} x^{8/3} y^{-1/6}}{x^{2/3} y^{1/3}} \\ &= 125^{2/3} x^2 y^{-1/2} \\ &= \frac{(\sqrt[3]{125})^2 x^2}{y^{1/2}} = \frac{25x^2 \cdot \sqrt{y}}{\sqrt{y} \sqrt{y}} \\ &= \frac{25x^2 \sqrt{y}}{y} \end{aligned}$$

62. $8^{-4/3}$

$$\begin{aligned} &\frac{1}{8^{4/3}} \\ &= \frac{1}{(\sqrt[3]{8})^4} \\ &= \frac{1}{2^4} \\ &= \boxed{\frac{1}{16}} \end{aligned}$$

63. $x^{-1/3}$

64. $x^{-3/4} \cdot x^{7/4}$

$$x^{\frac{4}{4}} = \boxed{x}$$

65. $x^{1/3} \cdot x^{1/4} \cdot x^{-1/2}$

$$\begin{aligned} &\frac{1}{3} + \frac{1}{4} + (-\frac{1}{2}) \\ &= x^{\frac{1}{12}} \\ &= \sqrt[12]{x} \end{aligned}$$