

6.5

The Remainder and Factor Theorems

What you should learn

GOAL 1 Divide polynomials and relate the result to the remainder theorem and the factor theorem.

GOAL 2 Use polynomial division in **real-life** problems, such as finding a production level that yields a certain profit in **Example 5**.

Why you should learn it

▼ To combine two **real-life** models into one new model, such as a model for money spent at the movies each year in **Ex. 62**.

**GOAL 1** DIVIDING POLYNOMIALS

When you divide a polynomial $f(x)$ by a divisor $d(x)$, you get a quotient polynomial $q(x)$ and a remainder polynomial $r(x)$. We write this as $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$. The degree of the remainder must be less than the degree of the divisor.

Example 1 shows how to divide polynomials using a method called **polynomial long division**.

EXAMPLE 1 Using Polynomial Long Division

Divide $2x^4 + 3x^3 + 5x - 1$ by $x^2 - 2x + 2$.

SOLUTION

Write division in the same format you would use when dividing numbers. Include a “0” as the coefficient of x^2 .

$$\begin{array}{r}
 \begin{array}{ccc}
 \frac{2x^4}{x^2} & \frac{7x^3}{x^2} & \frac{10x^2}{x^2} \\
 \downarrow & \downarrow & \downarrow \\
 2x^2 + & 7x + & 10
 \end{array} \\
 x^2 - 2x + 2 \overline{) 2x^4 + 3x^3 + 0x^2 + 5x - 1} \\
 \underline{2x^4 - 4x^3 + 4x^2} \quad \leftarrow \text{Subtract } 2x^2(x^2 - 2x + 2). \\
 7x^3 - 4x^2 + 5x \\
 \underline{7x^3 - 14x^2 + 14x} \quad \leftarrow \text{Subtract } 7x(x^2 - 2x + 2). \\
 10x^2 - 9x - 1 \\
 \underline{10x^2 - 20x + 20} \quad \leftarrow \text{Subtract } 10(x^2 - 2x + 2). \\
 11x - 21 \quad \leftarrow \text{remainder}
 \end{array}$$

At each stage, divide the term with the highest power in what's left of the dividend by the first term of the divisor. This gives the next term of the quotient.

Write the result as follows.

$$\frac{2x^4 + 3x^3 + 5x - 1}{x^2 - 2x + 2} = 2x^2 + 7x + 10 + \frac{11x - 21}{x^2 - 2x + 2}$$

✓ **CHECK** You can check the result of a division problem by multiplying the divisor by the quotient and adding the remainder. The result should be the dividend.

$$\begin{aligned}
 & (2x^2 + 7x + 10)(x^2 - 2x + 2) + 11x - 21 \\
 &= 2x^2(x^2 - 2x + 2) + 7x(x^2 - 2x + 2) + 10(x^2 - 2x + 2) + 11x - 21 \\
 &= 2x^4 - 4x^3 + 4x^2 + 7x^3 - 14x^2 + 14x + 10x^2 - 20x + 20 + 11x - 21 \\
 &= 2x^4 + 3x^3 + 5x - 1 \quad \checkmark
 \end{aligned}$$

ACTIVITY**Developing Concepts****Investigating Polynomial Division**

Let $f(x) = 3x^3 - 2x^2 + 2x - 5$.

- 1 Use long division to divide $f(x)$ by $x - 2$. What is the quotient? What is the remainder?
- 2 Use synthetic substitution to evaluate $f(2)$. How is $f(2)$ related to the remainder? What do you notice about the other constants in the last row of the synthetic substitution?

In the activity you may have discovered that $f(2)$ gives you the remainder when $f(x)$ is divided by $x - 2$. This result is generalized in the *remainder theorem*.

REMAINDER THEOREM

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

STUDENT HELP**Study Tip**

Notice that synthetic division could *not* have been used to divide the polynomials in Example 1 because the divisor, $x^2 - 2x + 2$, is not of the form $x - k$.

You may also have discovered in the activity that synthetic substitution gives the coefficients of the quotient. For this reason, synthetic substitution is sometimes called **synthetic division**. It can be used to divide a polynomial by an expression of the form $x - k$.

EXAMPLE 2 Using Synthetic Division

Divide $x^3 + 2x^2 - 6x - 9$ by (a) $x - 2$ and (b) $x + 3$.

SOLUTION

- a. Use synthetic division for $k = 2$.

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -6 & -9 \\ & & 2 & 8 & 4 \\ \hline & 1 & 4 & 2 & -5 \end{array}$$

$$\triangleright \frac{x^3 + 2x^2 - 6x - 9}{x - 2} = x^2 + 4x + 2 + \frac{-5}{x - 2}$$

- b. To find the value of k , rewrite the divisor in the form $x - k$.
Because $x + 3 = x - (-3)$, $k = -3$.

$$\begin{array}{r|rrrr} -3 & 1 & 2 & -6 & -9 \\ & & -3 & 3 & 9 \\ \hline & 1 & -1 & -3 & 0 \end{array}$$

$$\triangleright \frac{x^3 + 2x^2 - 6x - 9}{x + 3} = x^2 - x - 3$$

In part (b) of Example 2, the remainder is 0. Therefore, you can rewrite the result as:

$$x^3 + 2x^2 - 6x - 9 = (x^2 - x - 3)(x + 3)$$

This shows that $x + 3$ is a factor of the original dividend.

FACTOR THEOREM

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

Recall from Chapter 5 that the number k is called a *zero* of the function f because $f(k) = 0$.

EXAMPLE 3 Factoring a Polynomial

Factor $f(x) = 2x^3 + 11x^2 + 18x + 9$ given that $f(-3) = 0$.

SOLUTION

Because $f(-3) = 0$, you know that $x - (-3)$ or $x + 3$ is a factor of $f(x)$. Use synthetic division to find the other factors.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array}$$

The result gives the coefficients of the quotient.

$$\begin{aligned} 2x^3 + 11x^2 + 18x + 9 &= (x + 3)(2x^2 + 5x + 3) \\ &= (x + 3)(2x + 3)(x + 1) \end{aligned}$$

EXAMPLE 4 Finding Zeros of a Polynomial Function

One zero of $f(x) = x^3 - 2x^2 - 9x + 18$ is $x = 2$. Find the other zeros of the function.

SOLUTION

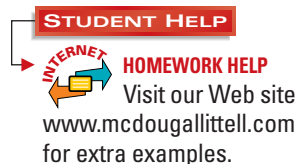
To find the zeros of the function, factor $f(x)$ completely. Because $f(2) = 0$, you know that $x - 2$ is a factor of $f(x)$. Use synthetic division to find the other factors.

$$\begin{array}{r|rrrr} 2 & 1 & -2 & -9 & 18 \\ & & 2 & 0 & -18 \\ \hline & 1 & 0 & -9 & 0 \end{array}$$

The result gives the coefficients of the quotient.

$$\begin{aligned} f(x) &= (x - 2)(x^2 - 9) && \text{Write } f(x) \text{ as a product of two factors.} \\ &= (x - 2)(x + 3)(x - 3) && \text{Factor difference of squares.} \end{aligned}$$

► By the factor theorem, the zeros of f are 2, -3 , and 3.



GOAL 2 USING POLYNOMIAL DIVISION IN REAL LIFE

In business and economics, a function that gives the price per unit p of an item in terms of the number x of units sold is called a *demand function*.

EXAMPLE 5 Using Polynomial Models

ACCOUNTING You are an accountant for a manufacturer of radios. The demand function for the radios is $p = 40 - 4x^2$ where x is the number of radios produced in millions. It costs the company \$15 to make a radio.

- Write an equation giving profit as a function of the number of radios produced.
- The company currently produces 1.5 million radios and makes a profit of \$24,000,000, but you would like to scale back production. What lesser number of radios could the company produce to yield the same profit?

PROBLEM SOLVING STRATEGY

SOLUTION

a. VERBAL MODEL

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$\text{Profit} = \text{Price per unit} \cdot \text{Number of units} - \text{Cost per unit} \cdot \text{Number of units}$$

LABELS

$$\begin{aligned} \text{Profit} &= P && \text{(millions of dollars)} \\ \text{Price per unit} &= 40 - 4x^2 && \text{(dollars per unit)} \\ \text{Number of units} &= x && \text{(millions of units)} \\ \text{Cost per unit} &= 15 && \text{(dollars per unit)} \end{aligned}$$

ALGEBRAIC MODEL

$$\begin{aligned} P &= (40 - 4x^2)x - 15x \\ P &= -4x^3 + 25x \end{aligned}$$

- Substitute 24 for P in the function you wrote in part (a).

$$24 = -4x^3 + 25x$$

$$0 = -4x^3 + 25x - 24$$

You know that $x = 1.5$ is one solution of the equation. This implies that $x - 1.5$ is a factor. So divide to obtain the following:

$$-2(x - 1.5)(2x^2 + 3x - 8) = 0$$

Use the quadratic formula to find that $x \approx 1.39$ is the other positive solution.

- The company can make the same profit by selling 1,390,000 units.

✓ **CHECK** Graph the profit function to confirm that there are two production levels that produce a profit of \$24,000,000.

FOCUS ON CAREERS



ACCOUNTANT

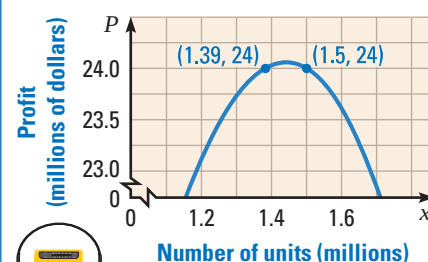
Most people think of accountants as working for many clients. However, it is common for an accountant to work for a single client, such as a company or the government.



CAREER LINK

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Radio Production



GUIDED PRACTICE

Vocabulary Check ✓
Concept Check ✓

- State the remainder theorem.
- Write a polynomial division problem that you would use long division to solve. Then write a polynomial division problem that you would use synthetic division to solve.
- Write the polynomial divisor, dividend, and quotient represented by the synthetic division shown at the right.

$$\begin{array}{r|rrrr} -3 & 1 & -2 & -9 & 18 \\ & & -3 & 15 & -18 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

Skill Check ✓

Divide using polynomial long division.

- $(2x^3 - 7x^2 - 17x - 3) \div (2x + 3)$
- $(-3x^3 + 4x - 1) \div (x - 1)$
- $(x^3 + 5x^2 - 2) \div (x + 4)$
- $(-x^3 + 2x^2 - 2x + 3) \div (x^2 - 1)$

Divide using synthetic division.

- $(x^3 - 8x + 3) \div (x + 3)$
- $(x^2 + 2x + 15) \div (x - 3)$
- $(x^4 - 16x^2 + x + 4) \div (x + 4)$
- $(x^2 + 7x - 2) \div (x - 2)$

Given one zero of the polynomial function, find the other zeros.

- $f(x) = x^3 - 8x^2 + 4x + 48$; 4
- $f(x) = 2x^3 - 14x^2 - 56x - 40$; 10
-  **BUSINESS** Look back at Example 5. If the company produces 1 million radios, it will make a profit of \$21,000,000. Find another number of radios that the company could produce to make the same profit.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice

to help you master skills is on p. 948.

USING LONG DIVISION Divide using polynomial long division.

- $(x^2 + 7x - 5) \div (x - 2)$
- $(2x^2 + 3x - 1) \div (x + 4)$
- $(x^2 + 5x - 3) \div (x - 10)$
- $(2x^4 + 7) \div (x^2 - 1)$
- $(6x^2 + x - 7) \div (2x + 3)$
- $(5x^4 + 14x^3 + 9x) \div (x^2 + 3x)$
- $(3x^2 + 11x + 1) \div (x - 3)$
- $(x^2 - 6x + 4) \div (x + 1)$
- $(x^3 - 3x^2 + x - 8) \div (x - 1)$
- $(x^3 + 8x^2 - 3x + 16) \div (x^2 + 5)$
- $(10x^3 + 27x^2 + 14x + 5) \div (x^2 + 2x)$
- $(2x^4 + 2x^3 - 10x - 9) \div (x^3 + x^2 - 5)$

USING SYNTHETIC DIVISION Divide using synthetic division.

- $(x^3 - 7x - 6) \div (x - 2)$
- $(4x^2 + 5x - 4) \div (x + 1)$
- $(2x^2 + 7x + 8) \div (x - 2)$
- $(x^2 + 10) \div (x + 4)$
- $(10x^4 + 5x^3 + 4x^2 - 9) \div (x + 1)$
- $(2x^4 - 6x^3 + x^2 - 3x - 3) \div (x - 3)$
- $(x^3 - 14x + 8) \div (x + 4)$
- $(x^2 - 4x + 3) \div (x - 2)$
- $(3x^2 - 10x) \div (x - 6)$
- $(x^2 + 3) \div (x + 3)$
- $(x^4 - 6x^3 - 40x + 33) \div (x - 7)$
- $(4x^4 + 5x^3 + 2x^2 - 1) \div (x + 1)$

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 15–26

Example 2: Exs. 27–38

Example 3: Exs. 39–46

Example 4: Exs. 47–54

Example 5: Exs. 60–62

FACTORING Factor the polynomial given that $f(k) = 0$.

39. $f(x) = x^3 - 5x^2 - 2x + 24$; $k = -2$ 40. $f(x) = x^3 - 3x^2 - 16x - 12$; $k = 6$
 41. $f(x) = x^3 - 12x^2 + 12x + 80$; $k = 10$ 42. $f(x) = x^3 - 18x^2 + 95x - 126$; $k = 9$
 43. $f(x) = x^3 - x^2 - 21x + 45$; $k = -5$ 44. $f(x) = x^3 - 11x^2 + 14x + 80$; $k = 8$
 45. $f(x) = 4x^3 - 4x^2 - 9x + 9$; $k = 1$ 46. $f(x) = 2x^3 + 7x^2 - 33x - 18$; $k = -6$

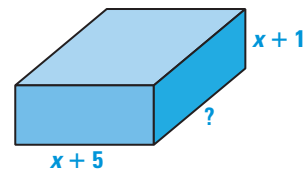
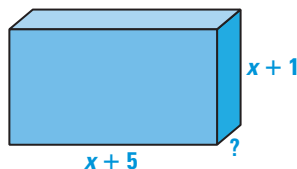
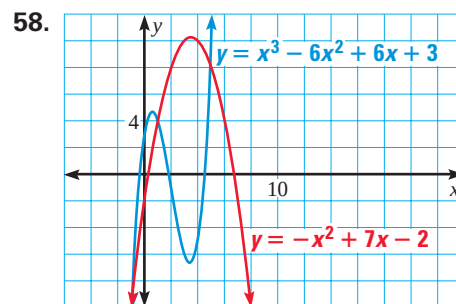
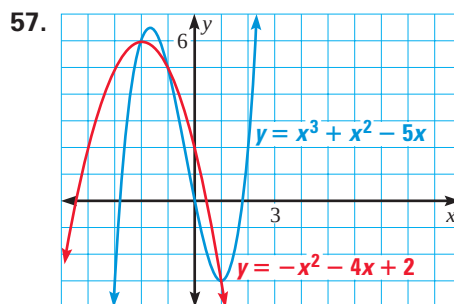
FINDING ZEROS Given one zero of the polynomial function, find the other zeros.

47. $f(x) = 9x^3 + 10x^2 - 17x - 2$; -2 48. $f(x) = x^3 + 11x^2 - 150x - 1512$; -14
 49. $f(x) = 2x^3 + 3x^2 - 39x - 20$; 4 50. $f(x) = 15x^3 - 119x^2 - 10x + 16$; 8
 51. $f(x) = x^3 - 14x^2 + 47x - 18$; 9 52. $f(x) = 4x^3 + 9x^2 - 52x + 15$; -5
 53. $f(x) = x^3 + x^2 + 2x + 24$; -3 54. $f(x) = 5x^3 - 27x^2 - 17x - 6$; 6

GEOMETRY CONNECTION You are given an expression for the volume of the rectangular prism. Find an expression for the missing dimension.

55. $V = 3x^3 + 8x^2 - 45x - 50$

56. $V = 2x^3 + 17x^2 + 40x + 25$

**POINTS OF INTERSECTION** Find all points of intersection of the two graphs given that one intersection occurs at $x = 1$.**59. LOGICAL REASONING** You divide two polynomials and obtain the result

$$5x^2 - 13x + 47 - \frac{102}{x+2}.$$

What is the dividend? How did you find it?

60. **COMPANY PROFIT** The demand function for a type of camera is given by the model $p = 100 - 8x^2$ where p is measured in dollars per camera and x is measured in millions of cameras. The production cost is \$25 per camera. The production of 2.5 million cameras yielded a profit of \$62.5 million. What other number of cameras could the company sell to make the same profit?

61. **FUEL CONSUMPTION** From 1980 to 1991, the total fuel consumption T (in billions of gallons) by cars in the United States and the average fuel consumption A (in gallons per car) can be modeled by

$$T = -0.026x^3 + 0.47x^2 - 2.2x + 72 \quad \text{and} \quad A = -8.4x + 580$$

where x is the number of years since 1980. Find a function for the number of cars from 1980 to 1991. About how many cars were there in 1990?

FOCUS ON APPLICATIONS**ALTERNATIVE FUEL**

Joshua and Kaia Tickell built the Green Grease Machine, which converts used restaurant vegetable oil into biodiesel fuel. The Tickells use the fuel in their motor home, the Veggie Van, as an alternative to the fuel referred to in Ex. 61.

**APPLICATION LINK**

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62. **MOVIES** The amount M (in millions of dollars) spent at movie theaters from 1989 to 1996 can be modeled by

$$M = -3.05x^3 + 70.2x^2 - 225x + 5070$$

where x is the number of years since 1989. The United States population P (in millions) from 1989 to 1996 can be modeled by the following function:

$$P = 2.61x + 247$$

Find a function for the average annual amount spent per person at movie theaters from 1989 to 1996. On average, about how much did each person spend at movie theaters in 1989? ▶ Source: *Statistical Abstract of the United States*

Test
Preparation

63. **MULTIPLE CHOICE** What is the result of dividing $x^3 - 9x + 5$ by $x - 3$?
- (A) $x^2 + 3x + 5$ (B) $x^2 + 3x$ (C) $x^2 + 3x + \frac{5}{x-3}$
- (D) $x^2 + 3x - \frac{5}{x-3}$ (E) $x^2 + 3x - 18 + \frac{59}{x-3}$

64. **MULTIPLE CHOICE** Which of the following is a factor of the polynomial $2x^3 - 19x^2 - 20x + 100$?

(A) $x + 10$ (B) $x + 2$ (C) $2x - 5$ (D) $x - 5$ (E) $2x + 5$

★ Challenge

65. **COMPARING METHODS** Divide the polynomial $12x^3 - 8x^2 + 5x + 2$ by $2x + 1$, $3x + 1$, and $4x + 1$ using long division. Then divide the same polynomial by $x + \frac{1}{2}$, $x + \frac{1}{3}$, and $x + \frac{1}{4}$ using synthetic division. What do you notice about the remainders and the coefficients of the quotients from the two types of division?

EXTRA CHALLENGE

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Mixed Review

CHECKING SOLUTIONS Check whether the given ordered pairs are solutions of the inequality. (Review 2.6)

66. $x + 7y \leq -8$; $(6, -2)$, $(-2, -3)$ 67. $2x + 5y \geq 1$; $(-2, 4)$, $(8, -3)$
68. $9x - 4y > 7$; $(-1, -4)$, $(2, 2)$ 69. $-3x - 2y < -6$; $(2, 0)$, $(1, 4)$

QUADRATIC FORMULA Use the quadratic formula to solve the equation. (Review 5.6 for 6.6)

70. $x^2 - 5x + 3 = 0$ 71. $x^2 - 8x + 3 = 0$ 72. $x^2 - 10x + 15 = 0$
73. $4x^2 - 7x + 1 = 0$ 74. $-6x^2 - 9x + 2 = 0$ 75. $5x^2 + x - 2 = 0$
76. $2x^2 + 3x + 5 = 0$ 77. $-5x^2 - x - 8 = 0$ 78. $3x^2 + 3x + 1 = 0$

POLYNOMIAL OPERATIONS Perform the indicated operation. (Review 6.3)

79. $(x^2 - 3x + 8) - (x^2 + x - 1)$ 80. $(14x^2 - 15x + 3) + (11x - 7)$
81. $(8x^3 - 1) - (22x^3 + 2x^2 - x - 5)$ 82. $(x + 5)(x^2 - x + 5)$

83. **CATERING** You are helping your sister plan her wedding reception. The guests have chosen whether they would like the chicken dish or the vegetarian dish. The caterer charges \$24 per chicken dish and \$21 per vegetarian dish. After ordering the dinners for the 120 guests, the caterer's bill comes to \$2766. How many guests requested chicken? (Lesson 3.2)