

Euclid's Axioms

We want to start with a handful of statements that are somehow obviously true, and then derive other, more complicated statements from these. The five statements that Euclid chose are, roughly:

1. Given two points, we can always draw a (unique) straight line segment connecting these points.
2. Given a straight line segment, we can (uniquely) extend the line segment indefinitely.
3. We can draw a (unique) circle with any desired center and radius.
4. All right angles are equal to one another.
5. *The parallel postulate.* If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Certainly the fifth axiom is fundamentally different from the first four. The first two axioms can obviously be verified with a ruler, the third needs a compass, and the fourth is quite clear on its own. The fifth axiom is much more complex, and is perhaps best explained by way of a picture. In Figure 1 we see that the angles α and β sum to some angle smaller than 180° — that is, they

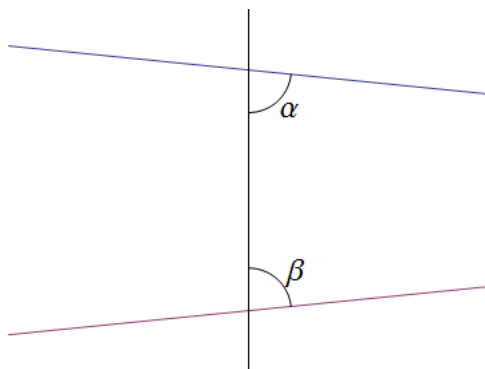


Figure 1: A line (black) transverse to a pair of lines.

are less than two right angles. Consequently, when we continue the two lines indefinitely, they intersect, and this intersection is on that side of the black line which has the smaller angle sum. Question: What can we now say *must* be true of parallel lines when sliced by a third line?

The relative complexity of the fifth axiom as compared to the first four raises some natural questions. Among them are:

1. Is the fifth axiom really true? It's not as obvious as the first four axioms, so can we find lines (or a model of geometry) that break it?

2. If the fifth axiom is true, do we have to take it as an axiom? Namely, can we prove the fifth axiom from the first four?
3. Can we replace the fifth axiom with something that is more obviously true?

Attempts have been made at answering these three questions essentially since the publication of Euclid's *Elements*, and the questions are obviously strongly related. A great deal of attention was especially paid to question 2, as can be seen [here](#) and [here](#). There was a relatively widely held belief that the fifth axiom, being so much more complex than the first four, should in fact be a theorem which can be derived from the first four axioms. In the early-to-mid 19th century, however, question 1 was answered, as mathematicians found [models of geometry which break the parallel postulate](#), but satisfy the first four axioms. This also answers question 2 in the negative: the first four axioms are true in these models, but the fifth is not.

Question 3 is an interesting and important one. The answer to this question is yes, and it turns out there are [many properties](#) which are logically equivalent to the parallel postulate in the presence of the first four axioms. Some of the equivalent properties that might be especially familiar to you include:

- The sum of the angles is the same for every triangle (namely, 180°).
- If three angles of a quadrilateral are right angles, then the fourth angle is also a right angle.
- In a right triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the two legs.
- In a plane, given a line and a point not on the line, at most one line parallel to the given line can be drawn through the given point.

The last equivalent property is known as [Playfair's axiom](#), and many treatments of Euclidean geometry use it in place of the parallel postulate. It is certainly less cumbersome (and perhaps a bit more self-evident) than the original parallel postulate.

To get a flavor of the proofs in Euclidean geometry, we're going to prove propositions 4, 5, and 6 of Euclid's *Elements* today. Our proofs will come (more or less) directly from *Elements*, and pictures will be provided in section. The first of these propositions might be known to you as the *side-angle-side* proposition.

Proposition 4. If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, the triangles are equal.

(*Proof.*) Let ABC and DEF be two triangles so that $\overline{AB} = \overline{DE}$, $\overline{AC} = \overline{DF}$, and $\angle BAC = \angle EDF$. We can lay the triangle ABC over the triangle DEF in such a way that the point A lies directly over the point D , and so that the line segment AB lies directly over the segment DE . Because $\overline{AB} = \overline{DE}$, this means that B lies directly over E . But since $\angle BAC = \angle EDF$, the line segment AC must coincide with DF , and thus C lies over F , since $\overline{AC} = \overline{DF}$. This means that the line segment BC connecting B to C is precisely the line segment EF connecting E to F . So the three sides of ABC coincide with the three sides of DEF , and thus the angles of these triangles also correspond. The triangles are equal, as desired. $\diamond$

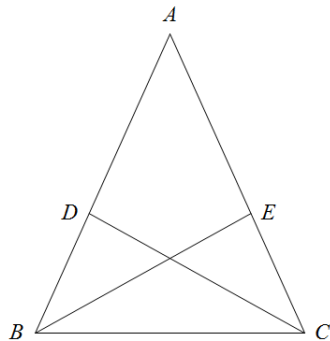


Figure 2: An isosceles triangle.

We can use this relatively simple proposition to prove the *pons asinorum*:

Proposition 5. The base angles of an isosceles triangle are equal to one another.

(*Proof.*) Suppose ABC is an isosceles triangle, with sides AB and AC of equal length (as seen in Figure 2), and choose an arbitrary point D along the side AB . Then AD is shorter than AB , hence shorter than AC , and we can choose a point E along AC so that $\overline{AD} = \overline{AE}$. Having chosen the points D and E , draw the segments BE and CD , as in Figure 2. Now we have

$$\overline{AB} = \overline{AC}, \quad \overline{AD} = \overline{AE}, \quad \text{and} \quad \angle BAE = \angle CAD.$$

According to proposition 4, this means that the triangles BAE and CAD are equivalent, and in particular $\overline{CD} = \overline{BE}$. Also, since $\overline{AB} = \overline{AC}$ and $\overline{AD} = \overline{AE}$, the remainders $\overline{DB}$ and $\overline{EC}$ are certainly equal. Now

$$\overline{CD} = \overline{BE}, \quad \overline{CB} = \overline{BC}, \quad \text{and} \quad \overline{DB} = \overline{EC}$$

tells us that the triangles CDB and BEC are equivalent. In particular this means that $\angle BCE = \angle CBD$. But then

$$\angle BCA = \angle BCE = \angle CBD = \angle CBA,$$

which is what we wanted to prove.

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The last proposition we'll prove today gives a sort of converse to proposition 5: it says that when a pair of angles is equal, so is a certain pair of sides. Its proof uses Proposition 4 — indicative of the manner in which this subject builds on itself — and also affords us an opportunity to write a proof by contradiction.

Proposition 6. If a triangle has two angles equal to one another, the sides which are opposite these angles are of equal length.

(*Proof.*) Let ABC be the angle in question, and suppose that $\angle ABC = \angle ACB$. For the sake of contradiction, we'll also assume that $\overline{AB} \neq \overline{AC}$. (Notice that AB is the side of ABC opposite $\angle ACB$, and AC is opposite $\angle ABC$.) In particular we assume that $\overline{AB} > \overline{AC}$ (if the inequality goes the other way, swap the labels B and C). Since $\overline{AB} > \overline{AC}$, we can choose a point D on the

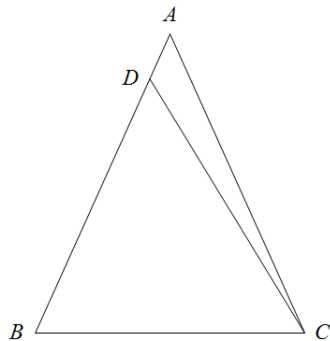


Figure 3: A triangle with $\angle ABC = \angle ACB$.

side AB so that $\overline{DB} = \overline{AC}$. We then draw the line segment DC , as in Figure 3. Then we have that

$$\overline{DB} = \overline{AC}, \quad \overline{BC} = \overline{CB}, \quad \text{and} \quad \angle DBC = \angle ABC = \angle ACB,$$

the last equality coming from our hypothesis. According to proposition 4, the triangles DBC and ACB coincide. But this is impossible, since DBC is contained in ACB . So we can't have $\overline{AB} > \overline{AC}$, meaning that we must have $\overline{AB} = \overline{AC}$. $\diamond$

Remark. With just a small modification, our proof of proposition 6 can be made into a *direct* proof — that is, we don't need to prove proposition 6 by contradiction. Can you figure out this modification?

It's interesting to note that none of our proofs used the parallel postulate. This means that our proofs work in any system of geometry where the first four axioms of Euclidean geometry hold, independent of whether or not the fifth postulate is true. Such statements belong to the purview of [absolute \(or neutral\) geometry](#), which uses only the first four axioms of Euclidean geometry. It's useful to point out which of our results fall into this category, since they will continue to hold when we study non-Euclidean geometries¹.

¹The non-Euclidean geometries we'll study satisfy the first four axioms