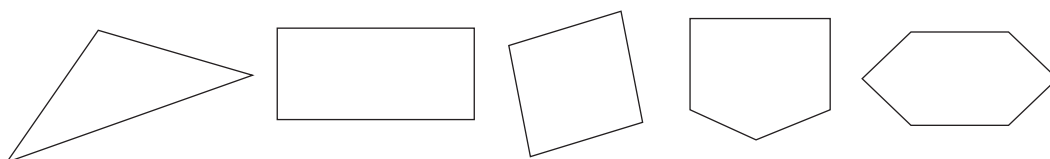


By studying this lesson you will be able to

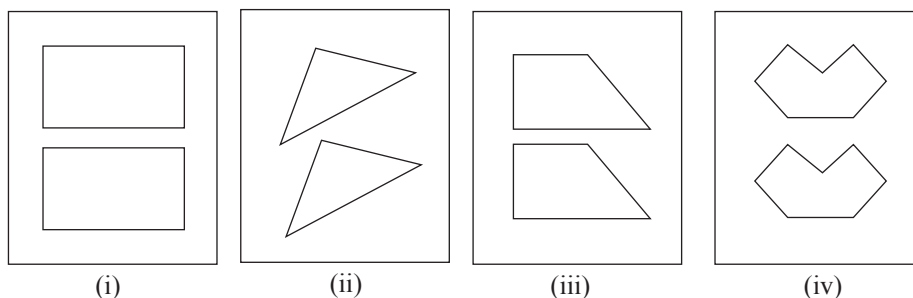
- recognize the congruence of two plane figures
- identify the necessary conditions for two triangles to be congruent
- prove riders by using the congruence of triangles

Congruence of two plane figures



If we examine the above figures we see that they are all closed plane figures consisting of straight line segments. Such figures are called rectilinear plane figures. The sides and the angles are called the elements (parts) of these figures.

The pairs of rectilinear plane figures presented below in figures (i) to (iv), which are identical in shape and size can be placed on each other such that they coincide.



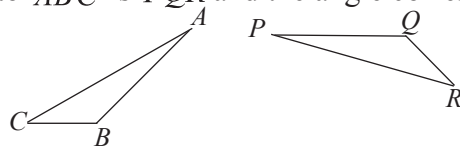
A pair of plane figures which may be made to coincide is called a pair of **congruent plane figures**. In this section we concentrate on the congruence of pairs of triangles.

5.1 Congruence of two triangles

A triangle has six elements. They are the three sides and the three angles of the triangle.

Let us assume that the two triangles ABC and PQR given below are congruent. Let us also assume that, when the two triangles are placed one on top of the other

such that they coincide, then AB coincides with PQ , AC coincides with PR and BC coincides with QR . Then we say that, in the two triangles, the side corresponding to AB is PQ , the side corresponding to AC is PR and the side corresponding to BC is QR . Similarly, we say that the angle corresponding to $\hat{BAC}$ is $\hat{QPR}$, the angle corresponding to $\hat{ABC}$ is $\hat{PQR}$ and the angle corresponding to $\hat{ACB}$ is $\hat{PRQ}$.



Accordingly, in congruent triangles, the corresponding elements are equal to each other.

We indicate the fact that two triangles are congruent by using the symbol " $\equiv$ ". For example, if the two triangles ABC and PQR are congruent, we indicate this fact by writing $\triangle ABC \equiv \triangle PQR$

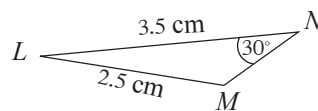
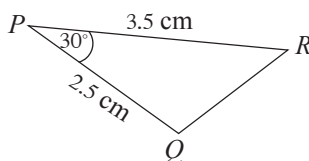
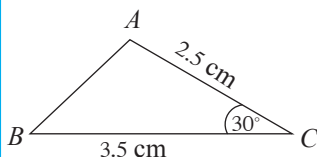
It is not necessary to show that the six elements of one triangle are equal to the six elements of the other triangle as indicated above to show that a pair of triangles is congruent. It is sufficient to show that three of the elements are equal. However, this does not mean that if any three elements of one triangle are equal to three elements of another triangle, then the two triangles are congruent. In certain cases, when three elements of one triangle are equal to three elements of another triangle, then the remaining elements too are equal and the triangles are congruent. There are four such cases. Let us now consider these four cases.

(a) First Case

The case in which two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle

Activity

Three triangles with two sides of length 2.5 cm and 3.5 cm and an angle of magnitude 30° are given below.



- Copy the triangle ABC onto a tissue paper and cut it out.
- Examine whether the cut out triangle can be made to coincide with the triangles PQR and LMN .
- Accordingly, select the triangle which is congruent to the triangle ABC .

It must be clear to you through the above activity, that only the triangle PQR is congruent to the triangle ABC . Both the triangles PQR and LMN have three elements which are equal to the given three elements of the triangle ABC . However, the triangle ABC is only congruent to the triangle PQR . It should be clear to you from this that, just because two triangles have three elements which are equal to each other, it does not mean that the triangles are congruent.

Let us consider another method by which we can identify that the triangle ABC is congruent to the triangle PQR but not to the triangle LMN . The 30° angle of the triangle ABC is the included angle of the sides of length 2.5 cm and 3.5 cm. It is the same for the triangle PQR . However the 30° angle is not the included angle of the sides of length 2.5 cm and 3.5 cm of the triangle LMN . Two sides and the included angle of triangle ABC are equal to two sides and the included angle of triangle PQR . But this cannot be said about the triangles ABC and LMN . $\therefore$ There is insufficient data to state that $\triangle ABC$ and $\triangle LMN$ are congruent.

Note: Here the angle $\hat{ACB}$ which is 30° is called the included angle of the sides AC and BC . Similarly, the angle $\hat{RPQ}$ is the included angle of the sides PR and PQ of the triangle PQR .

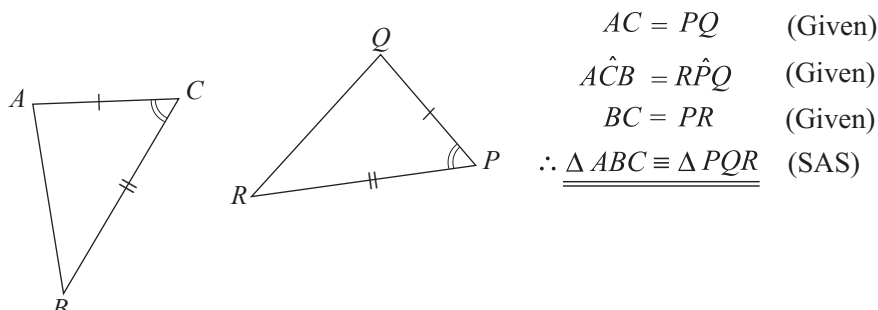
This result which you discovered through the above activity has been used as an axiom of geometry from ancient times.

If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, then the two triangles are congruent.

Showing that two triangles are congruent in the above manner is mentioned in short as being congruent under the case SAS.

The two triangles ABC and PQR given below can be shown to be congruent according to the above mentioned case using the given data as follows.

In the triangles ABC and PQR ,



$$\begin{aligned} AC &= PQ && \text{(Given)} \\ \hat{ACB} &= \hat{RPQ} && \text{(Given)} \\ BC &= PR && \text{(Given)} \\ \therefore \underline{\underline{\triangle ABC \equiv \triangle PQR}} &&& \text{(SAS)} \end{aligned}$$

Since the above two triangles are congruent, the remaining corresponding elements are also equal.

That is,

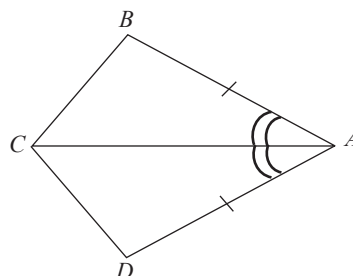
the sides AB and QR which are opposite the equal angles $\hat{ACB}$ and $\hat{QPR}$ are equal to each other, the angles $\hat{ABC}$ and $\hat{QRP}$ which are opposite the equal sides AC and PQ are equal to each other, and the angles $\hat{CAB}$ and $\hat{PQR}$ which are opposite the equal sides BC and PR are equal to each other.
Now let us consider an example.

Example 1

According to the data marked on the figure, prove that the triangles ABC and ADC are congruent, and write all the remaining equal corresponding elements.

Proof:

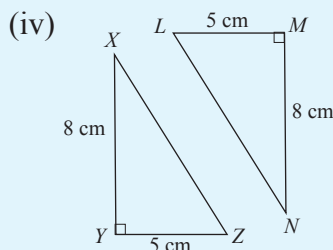
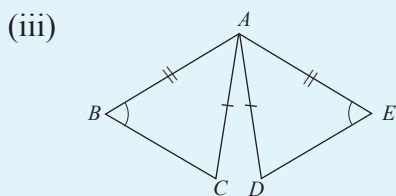
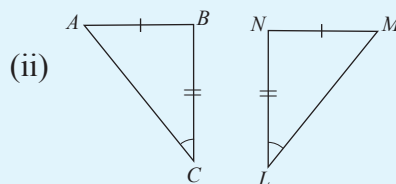
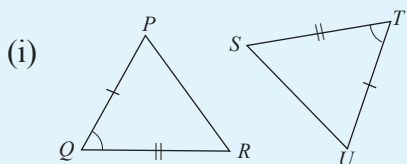
- (i) In the triangles ABC and ADC ,
 $AB = AD$ (Given)
 $\hat{BAC} = \hat{CAD}$ (Given)
 AC is a common side
 $\therefore \underline{\triangle ABC \equiv \triangle ADC}$ (S A S)

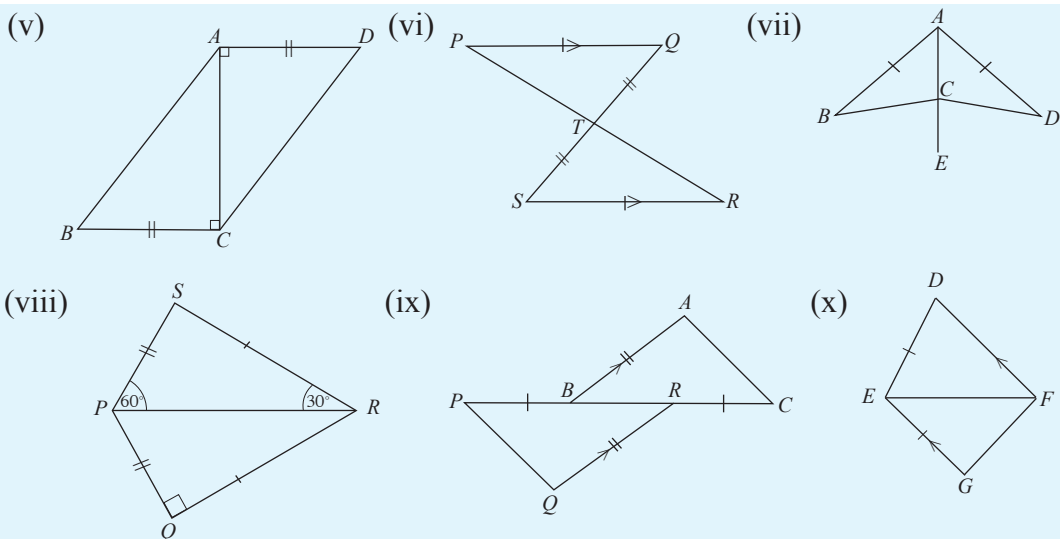


- (ii) The corresponding elements of congruent triangles are equal.
 $\therefore BC = CD$, $\hat{ABC} = \hat{ADC}$ and $\hat{ACB} = \hat{ACD}$.

Exercise 5.1

1. Determine for which pairs of triangles the SAS case can be applied to prove congruence based on the given information. For these pairs, prove that the two triangles are congruent and write down the remaining pairs of corresponding elements which are equal to each other.

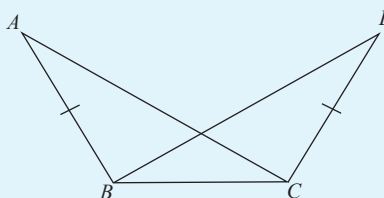




2. For each of the following parts, draw a sketch of the relevant pairs of triangles based on the information that is given. From these pairs of triangles, select the ones which are congruent and write down the remaining pairs of corresponding elements which are equal to each other.

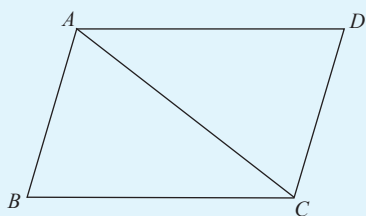
- In the triangles PQR and XYZ , $PQ = XZ$, $QR = XY$, $\hat{PQR} = \hat{YZX}$.
- In the triangles ABC and LMN , $AC = LN$, $BC = LM$, $\hat{ABC} = \hat{LMN} = 50^\circ$.
- In the triangles DEF and STU , $EF = TU$, $DF = SU$, $\hat{EFD} = \hat{TUS}$.
- In the triangles ABC and PQR , $BC = PQ$, $\hat{CBA} = \hat{QPR}$, $AC = PR$.

3. In the given figure, $AB = DC$ and $\hat{ABC} = \hat{BCD}$. Prove that,



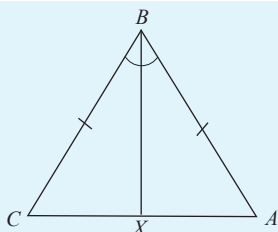
- $\triangle ABC \equiv \triangle DCB$,
- $AC = BD$.

4. In the quadrilateral $ABCD$, the sides AD and BC are parallel and equal in length. Mark the given data in the diagram and prove that,



- $\triangle ABC \equiv \triangle ADC$,
- $AB = DC$,
- AB and DC are parallel to each other.

5.



Using the information marked on the triangle ABC , prove that,

- (i) $\triangle ABX \equiv \triangle CBX$
- (ii) $\angle AXB = 90^\circ$.

6. In the quadrilateral $ABCD$, the diagonals AC and BD bisect each other at O .
Prove that,

- (i) $\triangle AOD \equiv \triangle BOC$
- (ii) the lines AD and BC are parallel to each other.

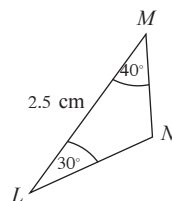
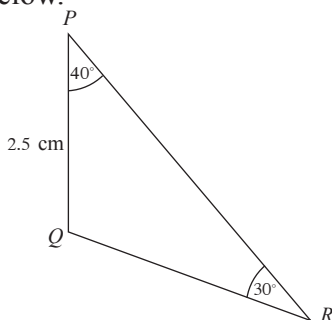
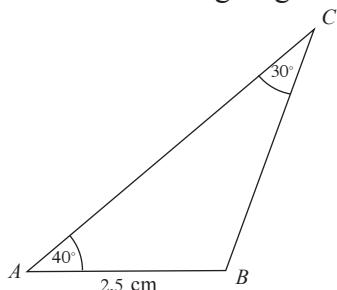
Now let us consider the second case by which the congruence of two triangles can be identified.

(b) Second Case

The case in which the magnitudes of two angles and the length of a side of a triangle are equal to the magnitudes of two angles and the length of a corresponding side of another triangle

Activity

Consider the triangles given below.



- Copy the triangle ABC onto a tissue paper and cut it out.
- Place it on the triangles PQR and LMN and examine which triangle it coincides with.
- Accordingly, which triangle is congruent to the triangle ABC ?

It must be clear to you according to this activity that the triangle ABC is congruent only to the triangle PQR .

In this case too, as in the case (a), the two triangles PQR and LMN have three elements which are equal to three elements of the triangle ABC . However, although the triangle ABC is congruent to the triangle PQR , it is not congruent to the triangle LMN . As before, it should be clear to you from this that, just because two triangles have three elements which are equal to each other, it does not mean that the triangles are congruent.

Therefore, let us consider another method by which we can identify that the triangle ABC is congruent to the triangle PQR . The given side of length 2.5 cm is opposite the given 30° angle of the triangle ABC . It is the same for the triangle PQR . However it is not the same in the triangle LMN . Accordingly, two angles of triangle ABC are equal to two angles of triangle PQR , and one side of triangle ABC is equal to the **corresponding** side of triangle PQR . However, the corresponding side of triangle LMN is not equal to that of triangle ABC .

Note: Here, corresponding sides are defined as those which are opposite equal angles of the two triangles.

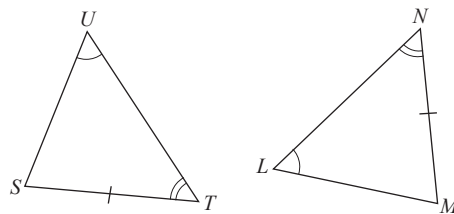
If two angles and a side of one triangle are equal to two angles and the corresponding side of another triangle, then the two triangles are congruent.

Showing that two triangles are congruent under these conditions is stated concisely as being congruent according to the AAS case.

Using the given data, we can show as done below, that the triangles STU and LMN in the following figure are congruent according to the case mentioned above.

In the triangles STU and LMN

$$\begin{aligned}\hat{S}TU &= \hat{M}NL \quad (\text{Given}) \\ \hat{T}US &= \hat{L}MN \quad (\text{Given}) \\ ST &= MN \quad (\text{Given}) \\ \therefore \triangle STU &\equiv \triangle MNL \quad (\text{AAS})\end{aligned}$$



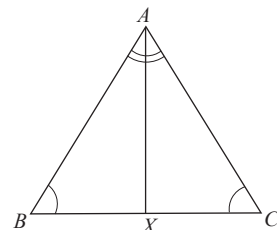
Note: In the above two triangles ST and MN are a pair of corresponding sides which are also equal. Observe carefully that they are corresponding sides because they are opposite the angles $\hat{S}UT$ and $\hat{M}LN$ which are equal to each other.

Example 1

Based on the data marked on the figure, prove that, $\triangle ABX \equiv \triangle ACX$ and write all the remaining equal corresponding elements.

Proof:

- (i) In the triangles ABX and ACX ,
- $$\begin{aligned}\hat{A}BX &= \hat{A}CX \quad (\text{Given}) \\ \hat{B}AX &= \hat{C}AX \quad (\text{Given}) \\ AX &\text{ is a common side} \\ \therefore \triangle ABX &\equiv \triangle ACX \quad (\text{AAS})\end{aligned}$$

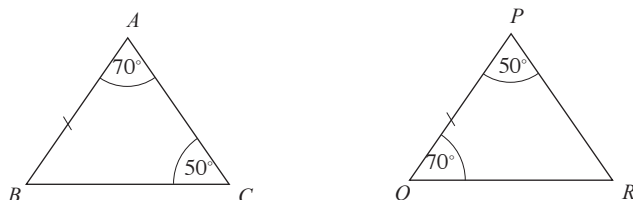


The corresponding elements of congruent triangles are equal.

Therefore $BX = CX$, $\hat{AXB} = \hat{AXC}$, $AB = AC$

Example 2

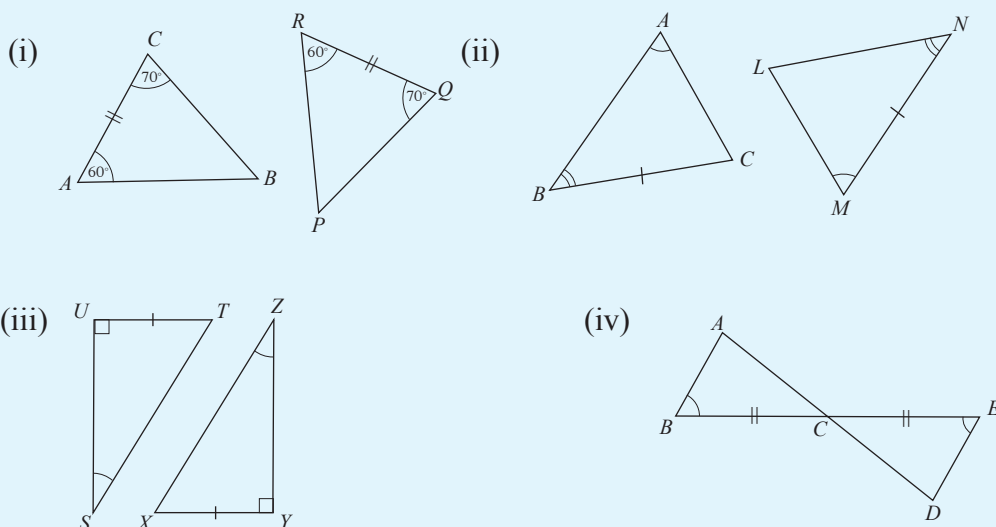
Determine whether the following pair of triangles is congruent under the case AAS.



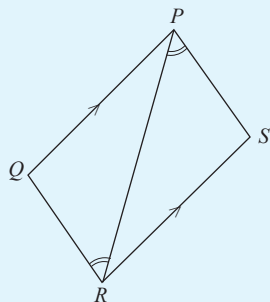
Two angles of the triangle ABC are equal to two angles of the triangle PQR . Also, $AB = PQ$. However, they are not corresponding sides. The reason for this is that the angles $\hat{ACB}$ and $\hat{PRQ}$ which are opposite these two sides are not equal to each other. ($\hat{ACB} = 50^\circ$, $\hat{PRQ} = 180^\circ - 50^\circ - 70^\circ = 60^\circ$) Therefore, there are insufficient reasons to say that the two triangles ABC and PQR are congruent to each other by the case AAS.

Exercise 5.2

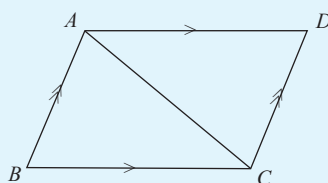
Mention for which pairs of triangles the AAS conditions can be applied to prove congruence based on the given information. For these pairs, prove that the two triangles are congruent and write down the other pairs of corresponding elements which are equal to each other.



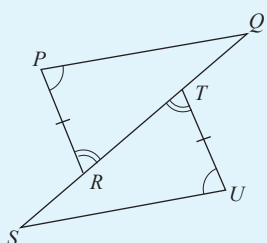
(v)



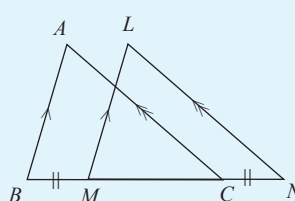
(vi)



(vii)



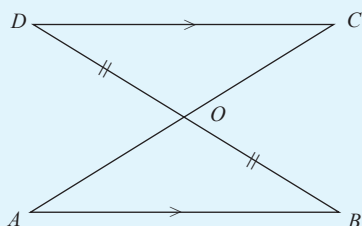
(viii)



2. For each of the following parts, draw a sketch of the relevant pair of triangles based on the information that is given. From these pairs of triangles, select the ones that are congruent and write down the remaining pairs of corresponding elements which are equal to each other.

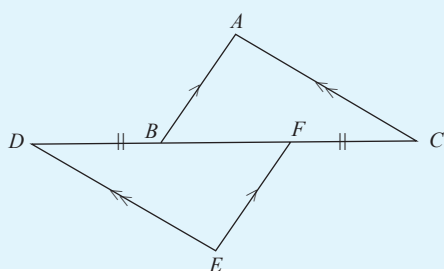
- In the triangles ABC and PQR , $\hat{A}BC = \hat{P}QR$, $\hat{A}CB = \hat{P}RQ$, $BC = QR$
- In the triangles XYZ and LMN , $\hat{X}YZ = \hat{L}MN = 90^\circ$, $\hat{Y}XZ = 30^\circ$, $\hat{M}NL = 60^\circ$, $YZ = MN$
- In the triangles STU and PQR , $\hat{T}SU = \hat{Q}RP$, $TU = PR$, $\hat{T}US = \hat{P}QR$
- In the triangles DEF and ABC , $\hat{E}DF = \hat{B}AC = 40^\circ$, $\hat{D}FE = \hat{A}CB = 60^\circ$, $DE = BA$

3.



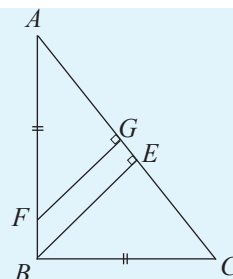
In the given figure, AB and CD are parallel to each other and $BO = OD$. Show that $\triangle AOB \equiv \triangle DOC$

4.



The pairs of sides AB and FE , and AC and DE are parallel to each other. Show that $\triangle ABC \equiv \triangle FED$

5. In the triangle ABC , $\hat{A} = 90^\circ$. If $AF = BC$, prove that $\triangle AFG \equiv \triangle BCE$.



6. In the quadrilateral $ABCD$, $\hat{A} = \hat{C} = 90^\circ$. The angles $\hat{ADC}$ and $\hat{ABC}$ are bisected by BD . Prove that $\triangle ABD \equiv \triangle CBD$.

Let us consider the third case by which the congruence of two triangles can be identified.

(c) Third Case

The case of three sides of a triangle being equal to three sides of another triangle

Can a unique triangle be constructed when the lengths of the three sides of the triangle are given? To determine this, engage in the following activity.

Activity

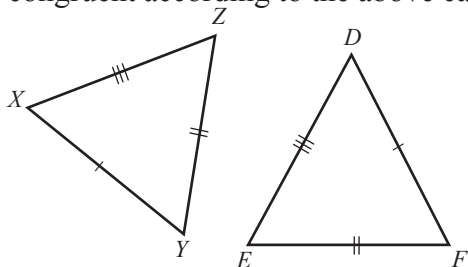
Break off pairs of ekel sticks of lengths 5 cm, 6 cm and 7 cm respectively. Make two triangles of side lengths 5 cm, 6 cm and 7 cm with these pieces. Do you see that the two triangles have to be congruent? By changing the positions of the pieces of ekel in one triangle, can you create a triangle which is not congruent to the other triangle? It must be clear to you that this is not possible.

This result which you established through the above activity can also be used as an axiom.

If the three sides of a triangle are equal to the three sides of another triangle, then the two triangles are congruent.

Showing that two triangles are congruent under these conditions is stated concisely as being congruent according to the SSS case.

We can show in the following manner that the pair of triangles XYZ and DEF are congruent according to the above case.



In the triangles XYZ and DEF ,

$$XY = DF \quad (\text{Given})$$

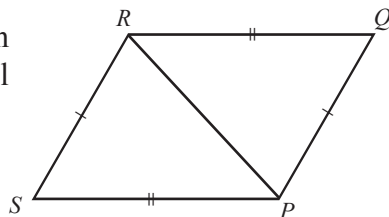
$$YZ = EF \quad (\text{Given})$$

$$ZX = DE \quad (\text{Given})$$

$$\therefore \triangle XYZ \equiv \triangle DEF \quad (\text{S S S})$$

Example 1

Prove that $\triangle PQR \equiv \triangle PSR$ based on the information in the figure and write all the remaining equal corresponding elements.



Proof: In the triangles PQR and PSR ,

$$PQ = RS \quad (\text{Given})$$

$$QR = PS \quad (\text{Given})$$

PR is a common side

$$\therefore \triangle PQR \equiv \triangle RSP \quad (\text{S S S})$$

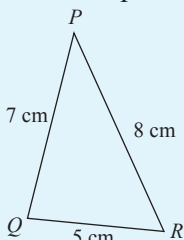
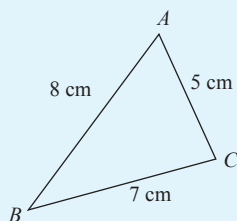
The corresponding elements of congruent triangle are equal

$$\therefore \angle RSP = \angle QPR, \angle SRP = \angle QRP, \angle SPR = \angle QRP.$$

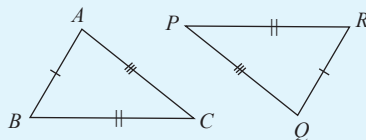
Exercise 5.3

1. Determine for which of the following pairs of the triangles the SSS conditions can be used to show congruence based on the given information. For these pairs, prove that the two triangles are congruent and write down the remaining pairs of corresponding elements which are equal to each other.

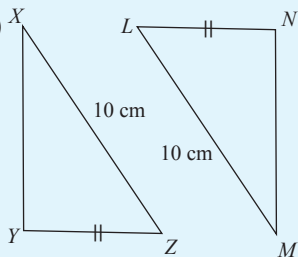
(i)



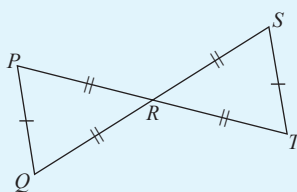
(ii)



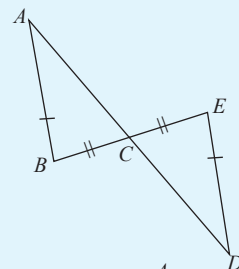
(iii)



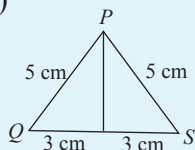
(iv)



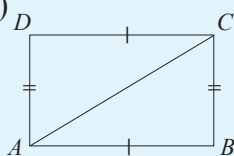
(v)



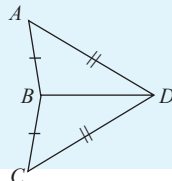
(vi)



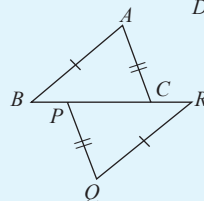
(vii)



(viii)



(ix)



2. Draw a sketch of the following triangles based on the information that is given. Select the triangles (if there are any) which are congruent according to the SSS case and write down the remaining pairs of corresponding elements which are equal to each other.

In triangle PQR , $PQ = 4$ cm, $QR = 6$ cm, $RP = 5$ cm

In triangle XYZ , $XY = 6$ cm, $YZ = 8$ cm, $ZX = 10$ cm

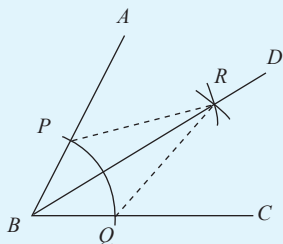
In triangle LMN , $LM = 5$ cm, $NM = 4$ cm, $NL = 6$ cm

In triangle DEF , $DE = 8$ cm, $EF = 10$ cm, $FD = 6$ cm

In triangle ABC , $BC = 8$ cm, $CA = 7$ cm, $AB = 9$ cm

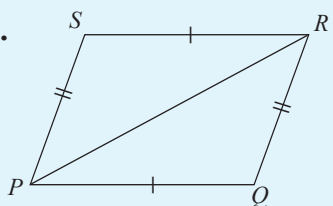
In triangle STU , $ST = 9$ cm, $TU = 7$ cm, $SU = 5$ cm

3.



To bisect the angle $\hat{ABC}$, a student selects the point B as the centre and draws the arc PQ . The arc intersects AB and BC at the points P and Q respectively. Two equal arcs drawn from the points P and Q intersect at R . Prove that $\hat{PBR} = \hat{QBR}$.

4.



The opposite sides of the quadrilateral $PQRS$ are equal in length. Prove that,

(i) $\triangle PSR \equiv \triangle PQR$

(ii) $\hat{PSR} = \hat{PQR}$

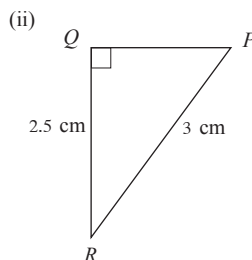
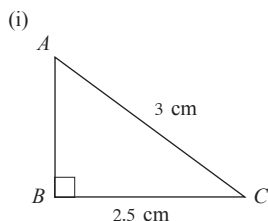
(iii) the opposite sides of the quadrilateral are parallel.

5. Prove that the straight line joining one vertex of an equilateral triangle to the mid-point of the opposite side is perpendicular to that side.

(d) Fourth Case

The case of the hypotenuse and a side of a right angled triangle being equal to the hypotenuse and a side of another right angled triangle.

A pair of right-angled triangles drawn such that the hypotenuse is 3 cm and another side is 2.5 cm is shown below.



Activity

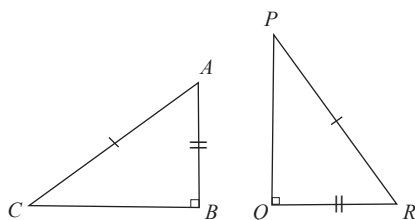
Copy the triangle in figure (i) onto a tissue paper and examine whether it can be made to coincide with the triangle in figure (ii).

Accordingly, the congruence of a pair of right-angled triangles can be expressed in terms of the equality of two elements as follows.

If the lengths of the hypotenuse and a side of a right-angled triangle are equal to the lengths of the hypotenuse and a side of another right-angled triangle, then the two triangles are congruent.

Showing that two triangles are congruent under these conditions is stated concisely as being congruent according to the RHS (right-angle-hypotenuse-side) case.

Let us prove that the two triangles given below are congruent based on the information that is given.



In the right-angled triangles ABC and PQR

$$AC = PR \quad (\text{Given})$$

$$AB = QR \quad (\text{Given})$$

$$\therefore \triangle ABC \equiv \triangle PQR \quad (\text{RHS})$$

Since the above pair of triangles is congruent, the remaining pairs of corresponding elements are also equal.

That is, $BC = PQ$, $\hat{BAC} = \hat{PRQ}$, $\hat{ACB} = \hat{QPR}$.

Example 1

Based on the information in the figure, prove that, $\triangle OXA \equiv \triangle OXB$ and write all the remaining equal corresponding elements.

Proof:

In the right-angled triangles OXA and OXB ,

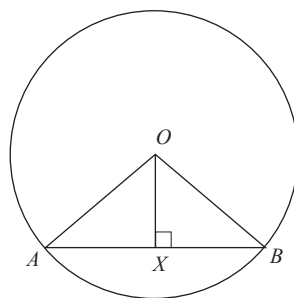
$$OA = OB \quad (\text{radii of same circle})$$

OX is a common side

$$\therefore \triangle OXA \equiv \triangle OXB \quad (\text{RHS})$$

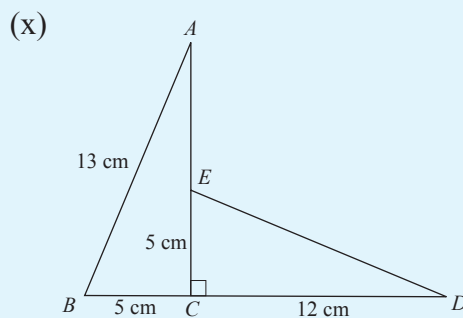
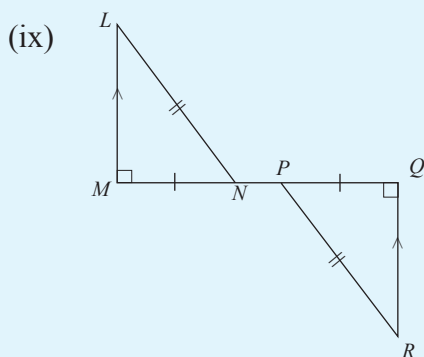
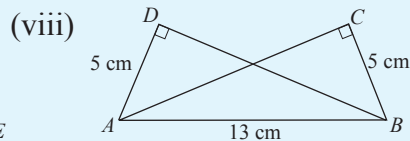
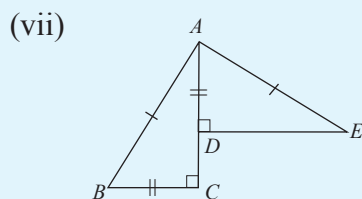
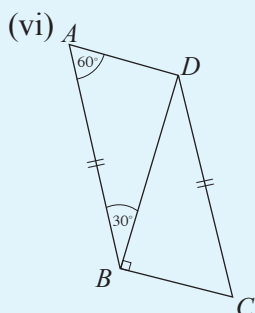
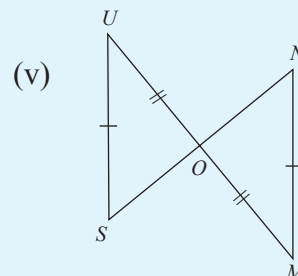
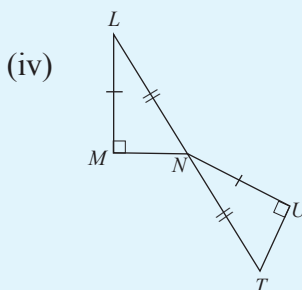
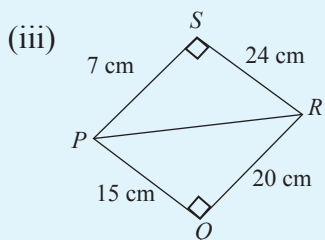
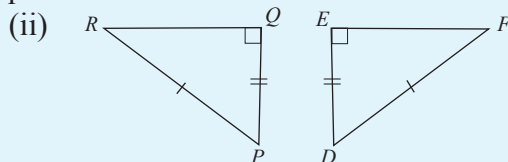
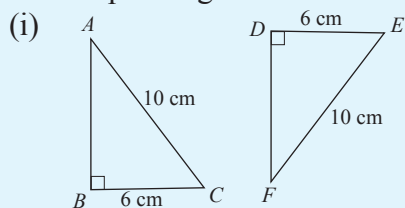
Corresponding elements of congruent triangles are equal.

$$\therefore \hat{OAX} = \hat{OBX}, AX = BX, \hat{AOX} = \hat{BOX}.$$



Exercise 5.4

1. Determine for which of the following pairs of the triangles the RHS conditions can be used to show congruence based on the given information. For these pairs, prove that the two triangles are congruent and write down the remaining pairs of corresponding elements which are equal to each other.



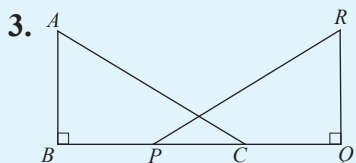
2. For each of the following parts, draw a sketch of the relevant pairs of triangles based on the information that is given. From these pairs, select the ones that are congruent (if there are any) and write down the remaining pairs of corresponding elements which are equal to each other.

(i) In the triangles ABC and PQR , $\hat{A}BC = \hat{P}QR = 90^\circ$, $AC = PR = 5$ cm, $BC = 3$ cm, $QP = 4$ cm.

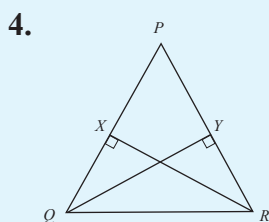
(ii) In the triangles LMN and XYZ , $\hat{L}MN = \hat{X}YZ = 90^\circ$, $LM = XY$, $MN = YZ$.

(iii) In the triangles DEF and PQR , $\hat{D}EF = \hat{P}QR = 90^\circ$, $DF = PR$, $\hat{F}DE = 20^\circ$, $\hat{P}RQ = 70^\circ$, $EF = PQ$.

(iv) In the triangles ABD and ABC , $\hat{A}DB = \hat{A}CB = 90^\circ$, $AD = CB$



If $AC = PR$ and $AB = RQ$ in the figure, show that $BP = CQ$.

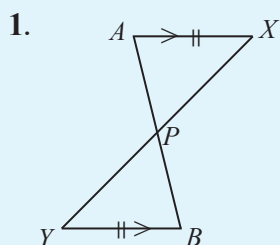


In the triangle PQR , the perpendiculars QY and RX are drawn from the points Q and R to the sides RP and QP respectively such that $QY = RX$. Prove that,

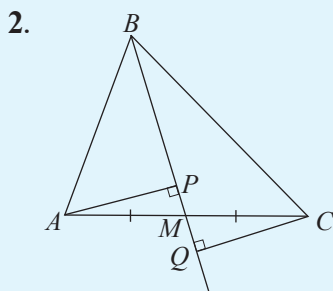
(i) $\triangle XQR \equiv \triangle YRQ$

(ii) $\hat{X}RQ = \hat{Y}QR$

Miscellaneous Exercise

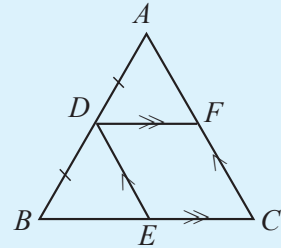


In the figure $AX \parallel YB$ and $AX = YB$. Show that the straight lines AB and YX bisect each other at P .

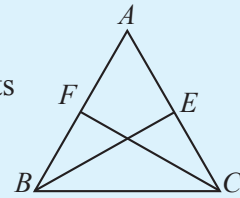


In the triangle ABC , the mid point of AC is M . The perpendiculars drawn from A and C meet BM and BM produced at P and Q respectively. Show that $\triangle AMP \equiv \triangle MQC$.

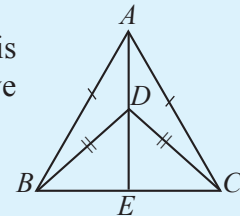
3. Using the information in the figure, show that $\triangle ADF \equiv \triangle DBE$.



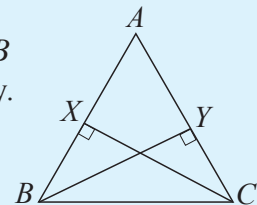
4. ABC in the figure is an equilateral triangle. The mid points of AC and AB are E and F respectively. Show that
 (i) AB and FC are perpendicular
 (ii) AC and BE are perpendicular
 (iii) $CF = BE$.



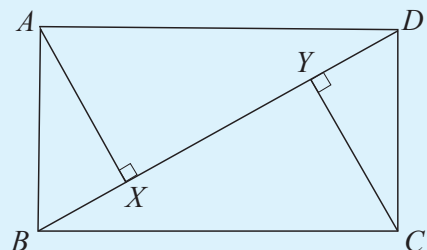
5. In the triangle ABC in the figure, $AB = AC$. The point D is such that $BD = DC$. AD produced meets BC at E . Prove that
 (i) $\triangle ABD \equiv \triangle CAD$
 (ii) $\triangle BAE \equiv \triangle CAE$
 (iii) AE and BC are perpendicular to each other.



6. In the given triangle ABC , the perpendicular drawn from B and C to the sides AC and AB are BY and CX respectively. If $BY = CX$ then show that
 (i) $AB = AC$
 (ii) $\angle XBC = \angle YCB$.

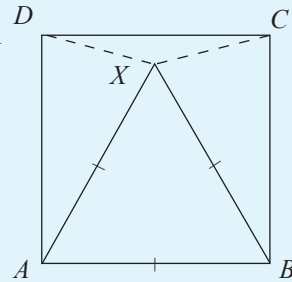


7. The perpendiculars drawn from A and C to the diagonal BD of the rectangle $ABCD$ meet BD at X and Y respectively. Prove that
 (i) $\triangle AXD \equiv \triangle BCY$
 (ii) $AX = YC$
 (iii) $BX = YD$
 (iv) $\triangle YDC \equiv \triangle ABX$



8. The point X lies in the interior of the square $ABCD$ such that XAB is an equilateral triangle. Show that

- (i) $\triangle AXD \equiv \triangle CBX$
- (ii) $\triangle DXC$ is an isosceles triangle.



9. The equilateral triangles BCF and DCE are drawn on the sides BC and DC of the square $ABCD$ such that the triangles lie outside the square.

- (i) Draw a rough sketch illustrating the above information.

Show that,

- (ii) $\triangle EDA \equiv \triangle ABF$
- (iii) $\triangle EAF$ is an equilateral triangle.

10. The perpendicular bisector of the side BC of the triangle ABC is AE . The point D lies on AE . Prove that,

- (i) $\triangle ABE \equiv \triangle AEC$
- (ii) $\triangle BDE \equiv \triangle DEC$
- (iii) $\triangle ABD \equiv \triangle ACD$

11. $ABCDE$ is a regular pentagon.

- (i) Show that $\triangle ABC \equiv \triangle AED$.
- (ii) If the foot of the perpendicular drawn from A to the side CD is X , prove that $CX = XD$.