

Similarity (geometry)

Two geometrical objects are called **similar** if they both have the same shape, or one has the same shape as the mirror image of the other. More precisely, one can be obtained from the other by uniformly scaling (enlarging or reducing), possibly with additional translation, rotation and reflection. This means that either object can be rescaled, repositioned, and reflected, so as to coincide precisely with the other object. If two objects are similar, each is congruent to the result of a particular uniform scaling of the other. A modern and novel perspective of similarity is to consider geometrical objects similar if one appears congruent to the other when zoomed in or out at some level.



Figures shown in the same color are similar

For example, all circles are similar to each other, all squares are similar to each other, and all equilateral triangles are similar to each other. On the other hand, ellipses are not all similar to each other, rectangles are not all similar to each other, and isosceles triangles are not all similar to each other

If two angles of a triangle have measures equal to the measures of two angles of another triangle, then the triangles are similar. Corresponding sides of similar polygons are in proportion, and corresponding angles of similar polygons have the same measure.

This article assumes that a scaling can have a scale factor of 1, so that all congruent shapes are also similar, but some school textbooks specifically exclude congruent triangles from their definition of similar triangles by insisting that the sizes must be different if the triangles are to qualify as similar

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Similar triangles

In geometry two triangles, $\triangle ABC$ and $\triangle A'B'C'$, are similar if and only if corresponding angles have the same measure: this implies that they are similar if and only if the lengths of corresponding sides are proportional.^[1] It can be shown that two triangles having congruent angles (*equiangular triangles*) are similar, that is, the corresponding sides can be proved to be proportional. This is known as the AAA similarity theorem.^[2] Due to this theorem, several authors simplify the definition of similar triangles to only require that the corresponding three angles are congruent.^[3]

There are several statements each of which is necessary and sufficient for two triangles to be similar:

- The triangles have two congruent angles^[4] which in Euclidean geometry implies that all their angles are congruent.^[5] That is:

If $\angle BAC$ is equal in measure to $\angle B'A'C'$, and $\angle ABC$ is equal in measure to $\angle A'B'C'$, then this implies that $\angle ACB$ is equal in measure to $\angle A'C'B'$ and the triangles are similar.

- All the corresponding sides have lengths in the same ratio^[6]

$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$. This is equivalent to saying that one triangle (or its mirror image) is an enlargement of the other.

- Two sides have lengths in the same ratio, and the angles included between these sides have the same measure^[7]
For instance:

$\frac{AB}{A'B'} = \frac{BC}{B'C'}$ and $\angle ABC$ is equal in measure to $\angle A'B'C'$.

This is known as the SAS similarity criterion.^[8]

When two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar, one writes^[9]: p. 22

$$\triangle ABC \sim \triangle A'B'C'.$$

There are several elementary results concerning similar triangles in Euclidean geometry.^[10]

- Any two equilateral triangles are similar.
- Two triangles, both similar to a third triangle are similar to each other (transitivity of similarity of triangles).
- Corresponding altitudes of similar triangles have the same ratio as the corresponding sides.
- Two right triangles are similar if the hypotenuse and one other side have lengths in the same ratio.^[11]

Given a triangle $\triangle ABC$ and a line segment $\overline{DE}$ one can, with ruler and compass find a point F such that $\triangle ABC \sim \triangle DEF$. The statement that the point F satisfying this condition exists is Wallis's postulate^[12] and is logically equivalent to Euclid's parallel postulate.^[13] In hyperbolic geometry (where Wallis's postulate is false) similar triangles are congruent.

In the axiomatic treatment of Euclidean geometry given by G.D. Birkhoff (see Birkhoff's axioms) the SAS similarity criterion given above was used to replace both Euclid's Parallel Postulate and the SAS axiom which enabled the dramatic shortening of Hilbert's axioms.^[8]

Other similar polygons

The concept of similarity extends to polygons with more than three sides. Given any two similar polygons, corresponding sides taken in the same sequence (even if clockwise for one polygon and counterclockwise for the other) are proportional and corresponding angles taken in the same sequence are equal in measure. However, proportionality of corresponding sides is not by itself sufficient to prove similarity for polygons beyond triangles (otherwise, for example, all rhombi would be similar). Likewise, equality of all angles in sequence is not sufficient to guarantee similarity (otherwise all rectangles would be similar). A sufficient condition for similarity of polygons is that corresponding sides and diagonals are proportional.

For given n , all regular n -gons are similar.

Similar curves

Several types of curves have the property that all examples of that type are similar to each other. These include:

- Circles
- Parabolas^[14]
- Hyperbolas of a specific eccentricity^[15]
- Ellipses of a specific eccentricity^[15]
- Catenaries^[16]
- Graphs of the logarithm function for different bases
- Graphs of the exponential function for different bases
- Logarithmic spirals

Similarity in Euclidean space

A **similarity** (also called a **similarity transformation** or **similitude**) of a Euclidean space is a bijection f from the space onto itself that multiplies all distances by the same positive real number r , so that for any two points x and y we have

$$d(f(x), f(y)) = rd(x, y),$$

where " $d(x, y)$ " is the Euclidean distance from x to y .^[17] The scalar r has many names in the literature including; the *ratio of similarity*, the *stretching factor* and the *similarity coefficient*. When $r = 1$ a similarity is called an isometry (rigid motion). Two sets are called **similar** if one is the image of the other under a similarity.

As a map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, a similarity of ratio r takes the form

$$f(x) = rAx + t,$$

where $A \in O_n(\mathbb{R})$ is an $n \times n$ orthogonal matrix and $t \in \mathbb{R}^n$ is a translation vector.

Similarities preserve planes, lines, perpendicularity, parallelism, midpoints, inequalities between distances and line segments.^[18] Similarities preserve angles but do not necessarily preserve orientation, *direct similitudes* preserve orientation and *opposite similitudes* change it.^[19]

The similarities of Euclidean space form a group under the operation of composition called the *similarities group* S .^[20] The direct similitudes form a normal subgroup of S and the Euclidean group $E(n)$ of isometries also forms a normal subgroup.^[21] The similarities group S is itself a subgroup of the affine group, so every similarity is an affine transformation.

One can view the Euclidean plane as the complex plane,^[22] that is, as a 2-dimensional space over the reals. The 2D similarity transformations can then be expressed in terms of complex arithmetic and are given by $f(z) = az + b$ (direct similitudes) and $f(z) = \bar{a}z + b$ (opposite similitudes), where a and b are complex numbers, $a \neq 0$. When $|a| = 1$, these similarities are isometries.

Ratios of sides, of areas, and of volumes

The ratio between the areas of similar figures is equal to the square of the ratio of corresponding lengths of those figures (for example, when the side of a square or the radius of a circle is multiplied by three, its area is multiplied by nine — i.e. by three squared). The altitudes of similar triangles are in the same ratio as corresponding sides. If a triangle has a side of length b and an altitude drawn to that side of length h then a similar triangle with corresponding side of length kb will have an altitude drawn to that side of length kh . The area of the first triangle is, $A = \frac{1}{2}bh$, while the area of the similar triangle will be $A' = \frac{1}{2}(kb)(kh) = k^2A$. Similar figures which can be decomposed into similar triangles will have areas related in the same way. The relationship holds for figures that are not rectifiable as well.

The ratio between the volumes of similar figures is equal to the cube of the ratio of corresponding lengths of those figures (for example, when the edge of a cube or the radius of a sphere is multiplied by three, its volume is multiplied by 27 — i.e. by three cubed).

Galileo's square–cube law concerns similar solids. If the ratio of similitude (ratio of corresponding sides) between the solids is k , then the ratio of surface areas of the solids will be k^2 , while the ratio of volumes will be k^3 .

Similarity in general metric spaces

In a general metric space (X, d) , an exact **similitude** is a function f from the metric space X into itself that multiplies all distances by the same positive scalar r , called f 's contraction factor, so that for any two points x and y we have

$$d(f(x), f(y)) = rd(x, y).$$

Weaker versions of similarity would for instance have f be a bi-Lipschitz function and the scalar r a limit

$$\lim \frac{d(f(x), f(y))}{d(x, y)} = r.$$

This weaker version applies when the metric is an effective resistance on a topologically self-similar set.

A self-similar subset of a metric space (X, d) is a set K for which there exists a finite set of similitudes $\{f_s\}_{s \in S}$ with contraction factors $0 \leq r_s < 1$ such that K is the unique compact subset of X for which

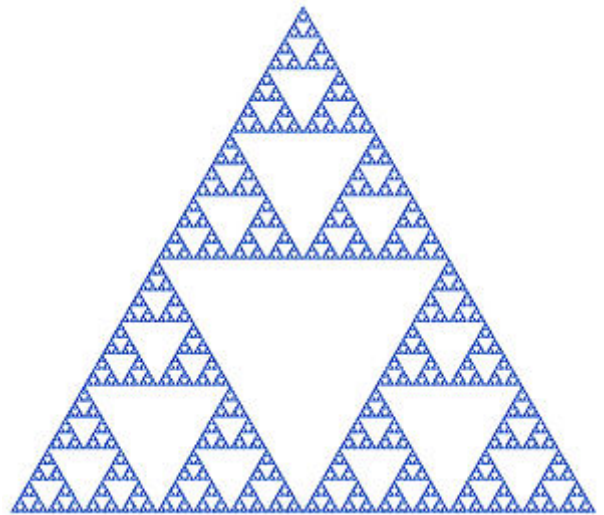
$$\bigcup_{s \in S} f_s(K) = K.$$

These self-similar sets have a self-similar measure μ^D with dimension D given by the formula

$$\sum_{s \in S} (r_s)^D = 1$$

which is often (but not always) equal to the set's Hausdorff dimension and packing dimension. If the overlaps between the $f_s(K)$ are "small", we have the following simple formula for the measure:

$$\mu^D(f_{s_1} \circ f_{s_2} \circ \cdots \circ f_{s_n}(K)) = (r_{s_1} \cdot r_{s_2} \cdots r_{s_n})^D.$$



Sierpiński triangle. A space having self-similarity dimension $\frac{\log 3}{\log 2} = \log_2 3$, which is approximately 1.58. (From Hausdorff dimension.)



A self-similar set constructed with two similitudes $z' = 0.1[(4+i)z+4]$ and $z' = 0.1[(4+7i)z^*+5-2i]$

Topology

In topology, a metric space can be constructed by defining a **similarity** instead of a distance. The similarity is a function such that its value is greater when two points are closer (contrary to the distance, which is a measure of **dissimilarity**: the closer the points, the lesser the distance).

The definition of the similarity can vary among authors, depending on which properties are desired. The basic common properties are

1. Positive defined:

$$\forall(a, b), S(a, b) \geq 0$$

2. Majored by the similarity of one element on itself (**auto-similarity**):

$$S(a, b) \leq S(a, a) \quad \text{and} \quad \forall(a, b), S(a, b) = S(a, a) \Leftrightarrow a = b$$

More properties can be invoked, such as **reflectivity** ($\forall(a, b) S(a, b) = S(b, a)$) or **finiteness** ($\forall(a, b) S(a, b) < \infty$). The upper value is often set at 1 (creating a possibility for a probabilistic interpretation of the similitude).

Self-similarity

Self-similarity means that a pattern is **non-trivially similar** to itself, e.g., the set $\{\dots, 0.5, 0.75, 1, 1.5, 2, 3, 4, 6, 8, 12, \dots\}$ of numbers of the form $\{2^i, 3 \cdot 2^i\}$ where i ranges over all integers. When this set is plotted on a logarithmic scale it has one-dimensional translational symmetry: adding or subtracting the logarithm of two to the logarithm of one of these numbers produces the logarithm of another of these numbers. In the given set of numbers themselves, this corresponds to a similarity transformation in which the numbers are multiplied or divided by two.

Psychology

The intuition for the notion of geometric similarity already appears in human children, as can be seen in their drawings^[23].

See also

- Congruence (geometry)
- Hamming distance (string or sequence similarity)
- Inversive geometry
- Jaccard index
- Proportionality
- Basic proportionality theorem
- Semantic similarity
- Similarity search
- Similarity space on Numerical taxonomy
- Homoeoid (shell of concentric, similar ellipsoids)
- Solution of triangles

Notes

1. Sibley 1998, p. 35
2. Stahl 2003, p. 127. This is also proved in Euclid's Elements, Book VI, Proposition 4.
3. For instance, Venema 2006, p. 122 and Henderson & Taimina 2005, p. 123
4. Euclid's elements Book VI Proposition 4.
5. This statement is not true in Non-euclidean geometry where the triangle angle sum is not 180 degrees.
6. Euclid's elements Book VI Proposition 5
7. Euclid's elements Book VI Proposition 6
8. Venema 2006, p. 143
9. Posamentier, Alfred S. and Lehmann, Ingmar *The Secrets of Triangles*, Prometheus Books, 2012.
10. Jacobs 1974, pp. 384 - 393
11. Hadamard, Jacques (2008), *Lessons in Geometry Vol. I: Plane Geometry* (<https://books.google.com/books?id=SaZwAAAAQBAJ&pg=PA125>), American Mathematical Society Theorem 120, p. 125, ISBN 9780821843673
12. Named for John Wallis (1616–1703)

13. Venema 2006, p. 122
14. a proof from academia.edu(https://www.academia.edu/5601461/Similarity_of_Parabolas_-_A_Geometrical_Perspective)
15. The shape of an ellipse or hyperbola depends only on the ratio b/a(<http://www.geom.uiuc.edu/docs/reference/CRC-formulas/node27.html>)
16. "Catenary" (http://xahlee.org/SpecialPlaneCurves_dir/Catenary_dir/catenary.html). Xahlee.org. 2003-05-28 Retrieved 2010-11-17.
17. Smart 1998, p. 92
18. Yale 1968, p. 47 Theorem 2.1
19. Pedoe 1988, pp. 179-181
20. Yale 1968, p. 46
21. Pedoe 1988, p. 182
22. This traditional term, as explained in its article, is a misnomer. This is actually the 1-dimensional complex line.
23. Cox, Dana Christine (2008). *Understanding Similarity: Bridging Geometric and Numeric Contexts for Proportional Reasoning* (<https://web.archive.org/web/20160601145518/https://books.google.com.br/books?id=-wpbd9wla4C&printsec=frontcover&hl=pt-BR#v=onepage&q&f=false>) (Ph.D.). ProQuest. ISBN 9780549756576 Archived from the original (<https://books.google.com.br/books?id=-wpbd9wla4C&printsec=frontcover&hl=pt-BR#v=onepage&q&f=false>) on 2016-06-01.

References

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Further reading

- Judith N. Cederberg (1989, 2001) *A Course in Modern Geometries* Chapter 3.12 Similarity Transformations, pp. 183–9, Springer ISBN 0-387-98972-2.
- H.S.M. Coxeter (1961, 9) *Introduction to Geometry*, §5 Similarity in the Euclidean Plane, pp. 67–76, §7 Isometry and Similarity in Euclidean Space, pp 96–104, John Wiley & Sons
- Günter Ewald (1971) *Geometry: An Introduction* pp 106, 181, Wadsworth Publishing
- George E. Martin (1982) *Transformation Geometry: An Introduction to Symmetry*, Chapter 13 Similarities in the Plane, pp. 136–46, Springer ISBN 0-387-90636-3.

External links

- Animated demonstration of similar triangles

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