

Section Formula



INTRODUCTION

In this chapter, we shall study section formula *i.e.* we shall find the coordinates of the point which divides (internally) the line segment joining two given points in a given ratio. We shall also find the coordinates of the mid-point of a line segment joining two given points and the coordinates of the centroid of a triangle whose vertices are given.

11.1 SECTION FORMULA

Let P, Q be two points in a plane (shown in the adjoining figure) and R be a point on the line segment joining points P and Q such that $PR : RQ = m_1 : m_2$, then we say that the point R divides the line segment PQ internally in the ratio $m_1 : m_2$.

Now, we shall develop a formula known as 'section formula' for finding the coordinates of R when we are given the coordinates of P and Q and the ratio in which R divides PQ internally.

Find the coordinates of the point which divides (internally) the line segment joining two given points in a given ratio.

Let P (x_1, y_1) and Q (x_2, y_2) be two given points in the coordinate plane, and R (x, y) be the point which divides PQ (internally) in the given ratio $m_1 : m_2$

$$\text{i.e. } \frac{PR}{RQ} = \frac{m_1}{m_2} \quad \dots(i)$$

Draw PM, QN and RL perpendiculars on x-axis, and through R draw a line parallel to x-axis to meet MP (produced) at S and NQ at T.

From the figure,

$$SR = ML = OL - OM = x - x_1 \quad \dots(ii)$$

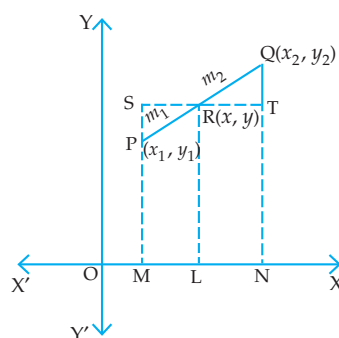
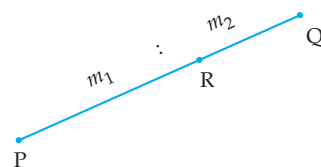
$$RT = LN = ON - OL = x_2 - x \quad \dots(iii)$$

$$PS = MS - MP = LR - MP = y - y_1 \quad \dots(iv)$$

$$TQ = NQ - NT = NQ - LR = y_2 - y \quad \dots(v)$$

Now ΔSPR is similar to ΔTQR

$$\therefore \frac{SR}{RT} = \frac{PR}{RQ}$$



(AA axiom of similarity)

$$\Rightarrow \frac{x - x_1}{x_2 - x} = \frac{m_1}{m_2} \quad (\text{using (ii), (iii) and (i)})$$

$$\Rightarrow m_2x - m_2x_1 = m_1x_2 - m_1x \Rightarrow m_1x + m_2x = m_1x_2 + m_2x_1$$

$$\Rightarrow (m_1 + m_2)x = m_1x_2 + m_2x_1 \Rightarrow x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}.$$

Again, as ΔSPR is similar to ΔTQR ,

$$\therefore \frac{PS}{TQ} = \frac{PR}{RQ}$$

$$\Rightarrow \frac{y - y_1}{y_2 - y} = \frac{m_1}{m_2} \quad (\text{using (iv), (v) and (i)})$$

$$\Rightarrow m_2y - m_2y_1 = m_1y_2 - m_1y \Rightarrow m_1y + m_2y = m_1y_2 + m_2y_1$$

$$\Rightarrow (m_1 + m_2)y = m_1y_2 + m_2y_1 \Rightarrow y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}.$$

Hence, the coordinates of R are $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2} \right)$.

Rule to write down the coordinates of the point which divides the line segment joining two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ internally in a given ratio $m_1 : m_2$.

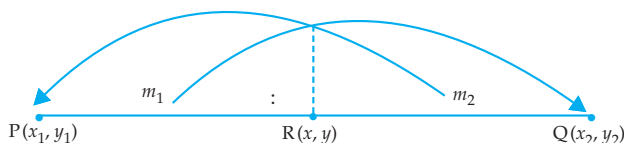
(i) Draw the line segment joining the given points P and Q.

(ii) Write down the coordinates of P and Q at extremities.

(iii) Let R (x, y) be the point which divides PQ internally in the ratio $m_1 : m_2$.

(iv) For x-coordinate of R, multiply m_1 with x_2 and m_2 with x_1 as shown in the figure given below by arrows and add the products. Divide the sum by $m_1 + m_2$. Thus $x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$.

(v) For y-coordinate of R, proceed as in step (iv).



11.1.1 Mid-point Formula

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two given points and $R(x, y)$ be the **mid-point** of the line segment PQ, then

$PR = RQ$, therefore, the ratio is $1 : 1$.

$$\therefore x = \frac{1 \cdot x_1 + 1 \cdot x_2}{1 + 1} \text{ and } y = \frac{1 \cdot y_1 + 1 \cdot y_2}{1 + 1}$$

$$\Rightarrow x = \frac{x_1 + x_2}{2} \text{ and } y = \frac{y_1 + y_2}{2}.$$

Hence, the coordinates of the mid-point of PQ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.

Remark

In problems where it is required to find the ratio when a given point divides the line segment joining two given points, it is convenient to take the ratio as $k : 1$, for, in this way two unknowns (m_1 and m_2) are reduced to one unknown and the section formula becomes

$$x = \frac{kx_2 + x_1}{k + 1} \text{ and } y = \frac{ky_2 + y_1}{k + 1}.$$

Then equate the abscissa or the ordinate of the point so obtained with that of the given point, and find the value of unknown k .

11.1.2 Centroid of a triangle

The point where the medians of a triangle meet is called the **centroid** of the triangle.

If AD is a median of the triangle ABC and G is its centroid, then $\frac{AG}{GD} = \frac{2}{1}$.

To find the centroid of a triangle whose vertices are given.

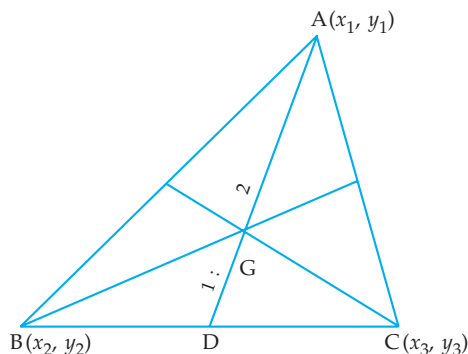
Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the given vertices of a triangle ABC. Let D be the mid-point of BC, then the coordinates of D are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$. Let G be the centroid of $\triangle ABC$, then G divides the median AD in the ratio 2 : 1. Therefore, coordinates of G are

$$\left(\frac{1 \cdot x_1 + 2 \cdot \frac{x_2 + x_3}{2}}{1 + 2}, \frac{1 \cdot y_1 + 2 \cdot \frac{y_2 + y_3}{2}}{1 + 2} \right) \text{ i.e. } \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right).$$

Remark

To prove that a quadrilateral is a

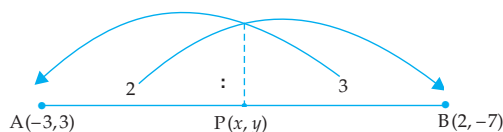
- (i) **Parallelogram**: show that diagonals bisect each other.
- (ii) **Rhombus**: show that diagonals bisect each other and two adjacent sides are equal.
- (iii) **Rectangle**: show that diagonals bisect each other and are equal.
- (iv) **Square**: show that diagonals bisect each other and two adjacent sides are equal and diagonals are also equal.



Illustrative Examples

Example 1. Calculate the coordinates of the point P which divides the line segment joining $A(-3, 3)$ and $B(2, -7)$ internally in the ratio 2 : 3.

Solution. Let (x, y) be the coordinates of the point P which divides the line segment joining $A(-3, 3)$ and $B(2, -7)$ internally in the ratio 2 : 3, then



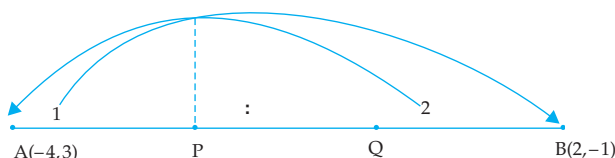
$$x = \frac{2 \times 2 + 3 \times (-3)}{2 + 3} = \frac{4 - 9}{5} = -\frac{5}{5} = -1 \text{ and}$$

$$y = \frac{2 \times (-7) + 3 \times 3}{2 + 3} = \frac{-14 + 9}{5} = -\frac{5}{5} = -1.$$

$\therefore$ The coordinates of P are $(-1, -1)$.

Example 2. Find the coordinates of the points of trisection of the line segment joining the points $A(-4, 3)$ and $B(2, -1)$.

Solution. Let P and Q be the points of trisection of the line segment AB, then



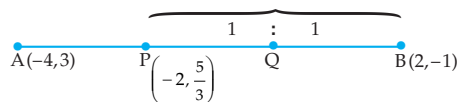
$$AP = PQ = QB \Rightarrow 2AP = PB$$

$$\Rightarrow \frac{AP}{PB} = \frac{1}{2} \Rightarrow P \text{ divides } AB \text{ in the ratio } 1 : 2.$$

$$\therefore \text{Coordinates of } P \text{ are } \left(\frac{1 \times 2 + 2 \times (-4)}{1+2}, \frac{1 \times (-1) + 2 \times 3}{1+2} \right) \text{ i.e. } \left(\frac{2-8}{3}, \frac{-1+6}{3} \right) \text{ i.e. } \left(-2, \frac{5}{3} \right).$$

Now $PQ = QB$

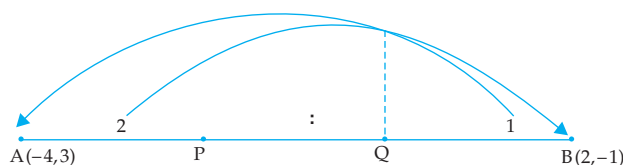
$\Rightarrow Q$ is the mid-point of PB .



$$\therefore \text{Coordinates of } Q \text{ are } \left(\frac{-2+2}{2}, \frac{\frac{5}{3}+(-1)}{2} \right) \text{ i.e. } \left(0, \frac{\frac{2}{3}}{2} \right) \text{ i.e. } \left(0, \frac{1}{3} \right).$$

Remark

Since $AP = PQ = QB$, $AQ = 2QB \Rightarrow Q$ divides AB in the ratio $2:1$.



$\therefore$ Coordinates of Q are

$$\left(\frac{2 \times 2 + 1 \times (-4)}{2+1}, \frac{2 \times (-1) + 1 \times 3}{2+1} \right) \text{ i.e. } \left(0, \frac{1}{3} \right).$$

Example 3. Find the coordinates of the points which divide the line segment joining the points $A(-2, 2)$ and $B(2, 8)$ into four equal parts.

Solution. Let P, Q and R be the points which divide the segment AB into four equal parts, then Q is mid-point of AB , P is mid-point of AQ and R is mid-point of QB .

$$\therefore Q \left(\frac{-2+2}{2}, \frac{2+8}{2} \right) \text{ i.e. } (0, 5),$$

$$P \left(\frac{-2+0}{2}, \frac{2+5}{2} \right) \text{ i.e. } \left(-1, \frac{7}{2} \right) \text{ and}$$

$$R \left(\frac{0+2}{2}, \frac{5+8}{2} \right) \text{ i.e. } \left(1, \frac{13}{2} \right)$$



Hence, the coordinates of the points of division are

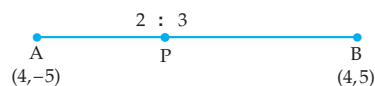
$$\left(-1, \frac{7}{2} \right), (0, 5), \left(1, \frac{13}{2} \right).$$

Example 4. If the line segment joining the points $A(4, -5)$ and $B(4, 5)$ is divided by the point P such that $\frac{AP}{AB} = \frac{2}{5}$, find the coordinates of P . (2007)

Solution. Given $\frac{AP}{AB} = \frac{2}{5} \Rightarrow 5AP = 2AB = 2(AP + PB)$

$$\Rightarrow 3AP = 2PB \Rightarrow \frac{AP}{PB} = \frac{2}{3}$$

$$\Rightarrow AP : PB = 2 : 3.$$



Thus, the point P divides the line segment joining the points A(4, -5) and B(4, 5) in the ratio 2 : 3 internally.

∴ The coordinates of P are $\left(\frac{2 \times 4 + 3 \times 4}{2 + 3}, \frac{2 \times 5 + 3 \times (-5)}{2 + 3}\right)$ i.e. (4, -1).

Example 5. Point A(4, -1) is reflected as A' in y-axis. Point B on reflection in x-axis is mapped as B'(-2, 5).

(i) Write the coordinates of A' and B.

(ii) Write the coordinates of the middle point of the line segment A'B.

Solution.

(i) Since the point A' is the reflection of the point A(4, -1) in the y-axis, the coordinates of A' are (-4, -1).

Also, as the point B'(-2, 5) is the reflection of the point B in the x-axis, the coordinates of B are (-2, -5).

(ii) The coordinates of the mid-point of A'B are $\left(\frac{-4 + (-2)}{2}, \frac{-1 + (-5)}{2}\right)$ i.e. (-3, -3).

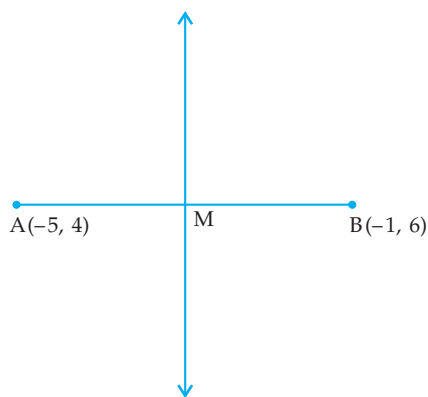
Example 6. Find a point which is equidistant from the points A(-5, 4) and B(-1, 6). How many such points are there?

Solution. Let M be the mid-point of the segment AB, then point M is equidistant from points A and B.

The point M is $\left(\frac{-5 + (-1)}{2}, \frac{4 + 6}{2}\right)$ i.e. (-3, 5).

Note that every point on the right bisector of the segment AB is equidistant from the points A and B.

Hence, the point (-3, 5) is equidistant from the given points A and B, and there are infinitely many points which are equidistant from points A and B.



Example 7. The mid-point of the line segment joining $(2a, 4)$ and $(-2, 2b)$ is $(1, 2a + 1)$. Find the values of a and b . (2007)

Solution. Let A, B be the points $(2a, 4)$, $(-2, 2b)$ respectively.

The coordinates of the mid-points of AB are

$$\left(\frac{2a + (-2)}{2}, \frac{4 + 2b}{2}\right) \text{ i.e. } (a - 1, 2 + b).$$

But the mid-point of AB is $(1, 2a + 1)$,

$$\therefore a - 1 = 1, \quad 2 + b = 2a + 1$$

$$\Rightarrow a = 2, \quad 2 + b = 2 \times 2 + 1 \Rightarrow a = 2, b = 3.$$

Hence, $a = 2$ and $b = 3$.

Example 8. The line segment joining the points A(2, 1) and B(5, -8) is trisected at the points P and Q such that P is nearer to A. If P lies on the line $2x - y + k = 0$, find the value of k .

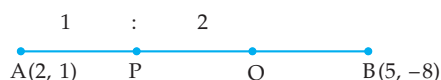
Solution. As AB is trisected at P and Q, and P is nearer to A, so AP : PB = 1 : 2.

$$\therefore P \left(\frac{1 \times 5 + 2 \times 2}{1 + 2}, \frac{1 \times (-8) + 2 \times 1}{1 + 2}\right) \text{ i.e. } P(3, -2)$$

Since P(3, -2) lies on $2x - y + k = 0$,

$$2 \times 3 - (-2) + k = 0$$

$$\Rightarrow 8 + k = 0 \Rightarrow k = -8.$$

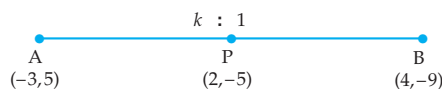


Example 9. In what ratio does the point $P(2, -5)$ divide the line segment joining the points $A(-3, 5)$ and $B(4, -9)$?

Solution. Let $P(2, -5)$ divide the line segment joining the points $A(-3, 5)$ and $B(4, -9)$ in the ratio $k : 1$ i.e. $AP : PB = k : 1$.

∴ Coordinates of P are

$$\left(\frac{k \times 4 + 1 \times (-3)}{k + 1}, \frac{k \times (-9) + 1 \times 5}{k + 1} \right)$$



But P is $(2, -5) \Rightarrow \frac{4k - 3}{k + 1} = 2, \frac{-9k + 5}{k + 1} = -5$.

$$\frac{4k - 3}{k + 1} = 2 \Rightarrow 4k - 3 = 2k + 2 \Rightarrow 2k = 5 \Rightarrow k = \frac{5}{2}.$$

Similarly $\frac{-9k + 5}{k + 1} = -5 \Rightarrow k = \frac{5}{2}$.

∴ The required ratio is $\frac{5}{2} : 1$ i.e. $5 : 2$ (internally).

Remark

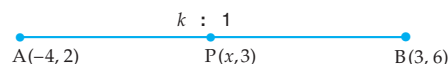
If we get different values of k from the two equations, it means that point P does not lie on the line AB, and the question of finding the ratio will not arise.

Example 10. Calculate the ratio in which the line joining $A(-4, 2)$ and $B(3, 6)$ is divided by point $P(x, 3)$. Also find (i) x (ii) Length of AP. (2014)

Solution. Let $P(x, 3)$ divide the line segment joining $A(-4, 2)$ and $B(3, 6)$ in the ratio $k : 1$ i.e. $AP : PB = k : 1$.

The point P is $\left(\frac{3k - 4}{k + 1}, \frac{6k + 2}{k + 1} \right)$.

But P is $(x, 3) \Rightarrow \frac{6k + 2}{k + 1} = 3$



$$\Rightarrow 6k + 2 = 3k + 3 \Rightarrow 3k = 1 \Rightarrow k = \frac{1}{3}.$$

∴ The required ratio is $\frac{1}{3} : 1$ i.e. $1 : 3$ (internally).

(i) As P is $(x, 3) \Rightarrow x = \frac{3k - 4}{k + 1}$, putting $k = \frac{1}{3}$, we get

$$x = \frac{3 \cdot \frac{1}{3} - 4}{\frac{1}{3} + 1} = \frac{-3}{\frac{4}{3}} = -\frac{9}{4}. \text{ Hence, } x = -\frac{9}{4}.$$

(ii) Length of AP = $\sqrt{\left(-\frac{9}{4} - (-4)\right)^2 + (3 - 2)^2}$ units = $\sqrt{\left(\frac{7}{4}\right)^2 + 1}$ units = $\frac{\sqrt{65}}{4}$ units.

Example 11. Determine the ratio in which the line $2x + y - 4 = 0$ divides the line segment joining the points $A(2, -2)$ and $B(3, 7)$.

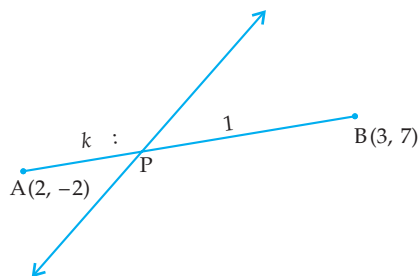
Solution. Given line is $2x + y - 4 = 0$ (i)

Let the given line (i) divide the line segment joining the points $A(2, -2)$ and $B(3, 7)$ in the ratio $k : 1$ at the point P i.e. $AP : PB = k : 1$.

∴ Coordinates of P are $\left(\frac{3k + 2}{k + 1}, \frac{7k - 2}{k + 1} \right)$.

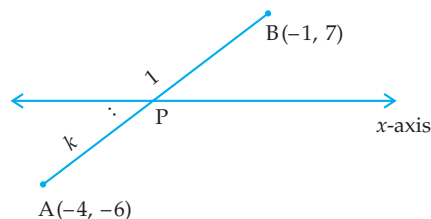
As the point P lies on the line (i), we have

$$\begin{aligned}
 2 \times \frac{3k+2}{k+1} + \frac{7k-2}{k+1} - 4 &= 0 \\
 \Rightarrow 2(3k+2) + (7k-2) - 4(k+1) &= 0 \\
 \Rightarrow 6k+4+7k-2-4k-4 &= 0 \\
 \Rightarrow 9k-2 &= 0 \Rightarrow k = \frac{2}{9} \\
 \therefore \text{The required ratio is } \frac{2}{9} : 1 \text{ i.e. } 2 : 9.
 \end{aligned}$$



Example 12. In what ratio does the x -axis divide the line segment joining the points $(-4, -6)$ and $(-1, 7)$? Also find the coordinates of the point of division.

Solution. Let the line segment joining the points $A(-4, -6)$ and $B(-1, 7)$ be divided by the x -axis in the ratio $k : 1$ at the point P i.e. $AP : PB = k : 1$, then the point P is $\left(\frac{-k-4}{k+1}, \frac{7k-6}{k+1}\right)$.



But P lies on the x -axis, therefore, y -coordinate of $P = 0$

$$\Rightarrow \frac{7k-6}{k+1} = 0 \Rightarrow 7k-6=0 \Rightarrow k = \frac{6}{7}.$$

$\therefore$ The required ratio is $\frac{6}{7} : 1$ i.e. $6 : 7$.

The coordinates of the point of division P are $\left(\frac{-\frac{6}{7}-4}{\frac{6}{7}+1}, 0\right)$ i.e. $\left(-\frac{34}{13}, 0\right)$.

Example 13. The line joining $P(-4, 5)$ and $Q(3, 2)$ intersects the y -axis at R . PM and QN are perpendiculars from P and Q on the x -axis. Find :

- the ratio $PR : RQ$.
- the coordinates of R .
- the area of the quadrilateral $PMNQ$.

Solution. (i) Let the line segment joining the points $P(-4, 5)$ and $Q(3, 2)$ be divided by the y -axis in the ratio $k : 1$ at R . By section formula, the coordinates of the point R are $\left(\frac{3k-4}{k+1}, \frac{2k+5}{k+1}\right)$. As the point R lies on the y -axis, x -coordinate of $R = 0$

$$\Rightarrow \frac{3k-4}{k+1} = 0 \Rightarrow 3k-4=0$$

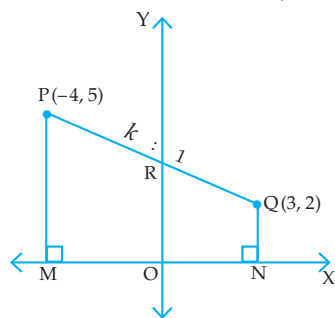
$$\Rightarrow k = \frac{4}{3}.$$

$\therefore$ The required ratio is $\frac{4}{3} : 1$ i.e. $4 : 3$.

(ii) The coordinates of the point R are $\left(0, \frac{2 \times \frac{4}{3} + 5}{\frac{4}{3} + 1}\right)$ i.e. $\left(0, \frac{23}{7}\right)$.

(iii) The quadrilateral $PMNQ$ is a trapezium.

$$\begin{aligned}
 \therefore \text{The area of quad. } PMNQ &= \frac{1}{2} (MP + NQ) \times MN \\
 &= \frac{1}{2} (5 + 2) \times 7 \text{ sq. units} = 24.5 \text{ sq. units.}
 \end{aligned}$$



(2004)

Example 14. If $P(9a - 2, -b)$ divides the line segment joining $A(3a + 1, -3)$ and $B(8a, 5)$ in the ratio $3 : 1$, find the values of a and b .

Solution. Given the point P divides the line segment joining points $A(9a - 2, -3)$ and $B(8a, 5)$ in the ratio $3 : 1$ i.e. $AP : PB = 3 : 1$, therefore, the point P is

$$\left(\frac{3 \times 8a + 1 \times (3a + 1)}{3 + 1}, \frac{3 \times 5 + 1 \times (-3)}{3 + 1} \right) \text{ i.e. } \left(\frac{27a + 1}{4}, 3 \right).$$

But the point P is $(9a - 2, -b)$

$$\Rightarrow 9a - 2 = \frac{27a + 1}{4} \text{ and } -b = 3$$

$$\Rightarrow 36a - 8 = 27a + 1 \text{ and } b = -3 \Rightarrow a = 1 \text{ and } b = -3.$$

Hence, $a = 1$ and $b = -3$.



Example 15. In the figure given below, the line segment AB meets X -axis at A and Y -axis at B . The point $P(-3, 4)$ on AB divides it in the ratio $2 : 3$. Find the coordinates of A and B . (2013)

Solution. Let the coordinates of the points A, B be $(a, 0), (0, b)$ respectively.

Given point $P(-3, 4)$ divides the segment AB in the ratio $2 : 3$

$$\text{i.e. } AP : PB = 2 : 3.$$

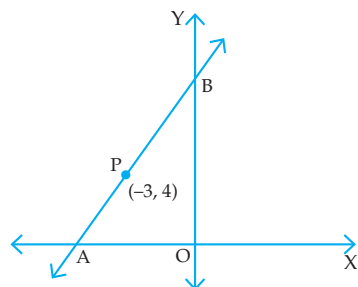
$\therefore$ Coordinates of P are

$$\left(\frac{2 \times 0 + 3 \times a}{2 + 3}, \frac{2 \times b + 3 \times 0}{2 + 3} \right) \text{ i.e. } \left(\frac{3a}{5}, \frac{2b}{5} \right).$$

$$\text{But } P \text{ is } (-3, 4) \Rightarrow \frac{3a}{5} = -3, \frac{2b}{5} = 4$$

$$\Rightarrow a = -5, b = 10.$$

$\therefore$ The coordinates of A and B are $(-5, 0)$ and $(0, 10)$ respectively.



Example 16. Find the reflection of the point $P(-3, 7)$ in the point $M(5, -1)$.

Solution. Let $P'(x, y)$ be the reflection of the point $P(-3, 7)$ in the point $M(5, -1)$, then M is the mid-point of the segment PP'

$$\Rightarrow \frac{-3 + x}{2} = 5, \frac{7 + y}{2} = -1$$

$$\Rightarrow -3 + x = 10, 7 + y = -2$$

$$\Rightarrow x = 13, y = -9.$$

$\therefore$ The reflection of the point P in the point M is the point $(13, -9)$.

Example 17. Find the length of the median through the vertex B of the triangle ABC with vertices $A(9, -2), B(-3, 7)$ and $C(-1, 10)$.

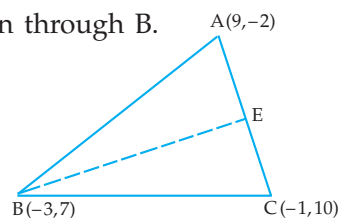
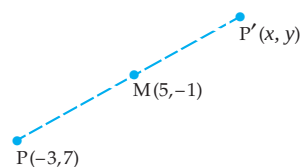
Solution. Let E be the mid-point of AC , then BE is the median through B .

As E is mid-point of AC , coordinates of E are

$$\left(\frac{9 + (-1)}{2}, \frac{-2 + 10}{2} \right) \text{ i.e. } (4, 4).$$

$$\therefore \text{Length of median } BE = \sqrt{(4 - (-3))^2 + (4 - 7)^2}$$

$$= \sqrt{(7)^2 + (-3)^2} = \sqrt{49 + 9} = \sqrt{58} \text{ units.}$$



Example 18. The centre of a circle is $C(-1, 6)$ and one end of a diameter is $A(5, 9)$. Find the coordinates of the other end.

Solution. Let the other end of the diameter of the circle be $B(x, y)$ whose one end is the point $A(5, 9)$.

∴ The mid-point of AB is $\left(\frac{5+x}{2}, \frac{9+y}{2}\right)$.

The centre of the circle is $C(-1, 6)$.

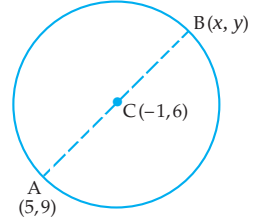
Since the centre of the circle is the mid-point of AB ,

$$\frac{5+x}{2} = -1 \text{ and } \frac{9+y}{2} = 6$$

$$\Rightarrow 5+x = -2 \text{ and } 9+y = 12$$

$$\Rightarrow x = -7 \text{ and } y = 3.$$

∴ The coordinates of the other end of the diameter are $(-7, 3)$.



Example 19. Prove by section formula that the points $(10, -6)$, $(2, -6)$, $(-4, -2)$ and $(4, -2)$, taken in this order, are the vertices of a parallelogram.

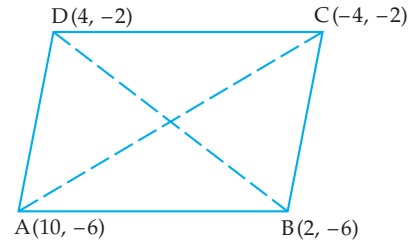
Solution. Let A, B, C and D be the points $(10, -6)$, $(2, -6)$, $(-4, -2)$, and $(4, -2)$ respectively.

We know that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.

Here, mid-point of diagonal AC is $\left(\frac{10+(-4)}{2}, \frac{(-6)+(-2)}{2}\right)$

i.e. $(3, -4)$ and mid-point of diagonal BD is $\left(\frac{2+4}{2}, \frac{(-6)+(-2)}{2}\right)$
i.e. $(3, -4)$.

Thus, the diagonals AC and BD have same mid-point, so the diagonals AC and BD bisect each other. Therefore, $ABCD$ is a parallelogram.



Example 20. Three consecutive vertices of a parallelogram $ABCD$ are $A(10, -6)$, $B(2, -6)$ and $C(-4, -2)$, find the fourth vertex D .

Solution. In the parallelogram $ABCD$, the three consecutive vertices are $A(10, -6)$, $B(2, -6)$ and $C(-4, -2)$. Let the fourth vertex D be (x, y) .

The mid-point of AC is $\left(\frac{10+(-4)}{2}, \frac{-6+(-2)}{2}\right)$
i.e. $(3, -4)$... (i)

The mid-point of BD is

$$\left(\frac{2+x}{2}, \frac{-6+y}{2}\right) \quad \dots (ii)$$

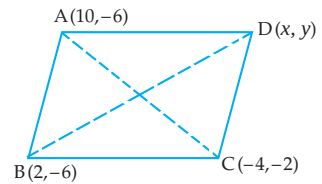
Since the diagonals of a parallelogram bisect each other, the mid-points of AC and BD are same.

∴ From (i) and (ii), $\frac{2+x}{2} = 3$ and $\frac{-6+y}{2} = -4$

$$\Rightarrow 2+x = 6 \text{ and } -6+y = -8$$

$$\Rightarrow x = 4 \text{ and } y = -2.$$

∴ The fourth vertex of the parallelogram $ABCD$ is $D(4, -2)$.



Example 21. The mid-points D, E and F of the sides AB, BC and CA of a triangle are $(3, 4)$, $(8, 9)$ and $(6, 7)$ respectively. Find the coordinates of the vertices of the triangle.

Solution. Let the vertices A, B and C of the triangle ABC be (x_1, y_1) , (x_2, y_2) and (x_3, y_3) respectively.

Since points D and F are mid-points of the sides AB and AC respectively, by Mid-point Theorem, $DF \parallel BC$ and $DF = \frac{1}{2} BC$ but E is mid-point of BC, so $DF \parallel BE$ and $DF = BE$.

Therefore, DBEF is a parallelogram.

Similarly, DECF and DEFA are parallelograms.

Since the diagonals of a parallelogram bisect each other, mid-points of diagonals BF and DE are same.

$$\therefore \frac{x_2 + 6}{2} = \frac{3 + 8}{2} \text{ and } \frac{y_2 + 7}{2} = \frac{4 + 9}{2}$$

$$\Rightarrow x_2 + 6 = 11 \text{ and } y_2 + 7 = 13 \Rightarrow x_2 = 5 \text{ and } y_2 = 6.$$

Therefore, the point B is (5, 6).

As DECF is a parallelogram, mid-points of CD and EF are same.

$$\therefore \frac{x_3 + 3}{2} = \frac{8 + 6}{2} \text{ and } \frac{y_3 + 4}{2} = \frac{9 + 7}{2}$$

$$\Rightarrow x_3 + 3 = 14 \text{ and } y_3 + 4 = 16.$$

$$\Rightarrow x_3 = 11 \text{ and } y_3 = 12.$$

Therefore, the point C is (11, 12).

Also DEFA is a parallelogram, mid-points of AE and DF are same.

$$\therefore \frac{x_1 + 8}{2} = \frac{3 + 6}{2} \text{ and } \frac{y_1 + 9}{2} = \frac{4 + 7}{2}$$

$$\Rightarrow x_1 + 8 = 9 \text{ and } y_1 + 9 = 11$$

$$\Rightarrow x_1 = 1 \text{ and } y_1 = 2.$$

Therefore, the point A is (1, 2).

Hence, the vertices of the triangle are A(1, 2), B(5, 6) and C(11, 12).

Alternative method

Let A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) be the vertices of the triangle ABC such that D(3, 4), E(8, 9) and F(6, 7) are mid-points of the sides AB, BC and CA respectively.

Since D(3, 4) is the mid-point of AB,

$$\therefore \frac{x_1 + x_2}{2} = 3 \text{ and } \frac{y_1 + y_2}{2} = 4$$

$$\Rightarrow x_1 + x_2 = 6 \quad \dots(i) \quad \text{and} \quad y_1 + y_2 = 8 \quad \dots(ii)$$

As E(8, 9) is the mid-point of BC,

$$\therefore \frac{x_2 + x_3}{2} = 8 \text{ and } \frac{y_2 + y_3}{2} = 9$$

$$\Rightarrow x_2 + x_3 = 16 \quad \dots(iii) \quad \text{and} \quad y_2 + y_3 = 18 \quad \dots(iv)$$

Also F(6, 7) is the mid-point of AC,

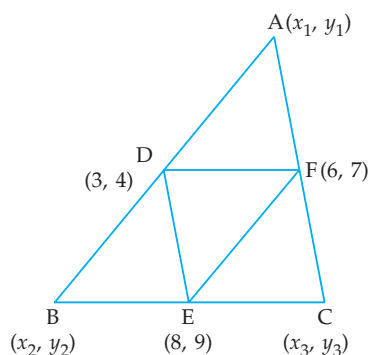
$$\therefore \frac{x_3 + x_1}{2} = 6 \text{ and } \frac{y_3 + y_1}{2} = 7$$

$$\Rightarrow x_3 + x_1 = 12 \quad \dots(v) \quad \text{and} \quad y_3 + y_1 = 14 \quad \dots(vi)$$

Adding (i), (iii) and (v), we get

$$2(x_1 + x_2 + x_3) = 34 \Rightarrow x_1 + x_2 + x_3 = 17 \quad \dots(vii)$$

Subtracting (i), (iii) and (v) from (vii) in turn, we get



$$x_3 = 11, x_1 = 1 \text{ and } x_2 = 5.$$

Adding (ii), (iv) and (vi), we get

$$2(y_1 + y_2 + y_3) = 40 \Rightarrow y_1 + y_2 + y_3 = 20 \quad \dots(viii)$$

Subtracting (ii), (iv) and (vi) from (viii) in turn, we get

$$y_3 = 12, y_1 = 2 \text{ and } y_2 = 6.$$

Hence, the vertices of the triangle are A(1, 2), B(5, 6) and C(11, 12).

Example 22. Two vertices of a triangle are $(-1, 4)$ and $(5, 2)$. If the centroid is $(0, -3)$, find the third vertex.

Solution. Two vertices of a triangle are $(-1, 4)$ and $(5, 2)$. Let the third vertex be (x, y) , then the centroid of the triangle is

$$\left(\frac{-1+5+x}{3}, \frac{4+2+y}{3} \right) \quad \left| \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3} \right) \right.$$

$$\text{i.e.} \left(\frac{x+4}{3}, \frac{y+6}{3} \right).$$

But the centroid of the triangle is $(0, -3)$

$$\Rightarrow \frac{x+4}{3} = 0 \text{ and } \frac{y+6}{3} = -3$$

$$\Rightarrow x + 4 = 0 \text{ and } y + 6 = -9$$

$$\Rightarrow x = -4 \text{ and } y = -15.$$

$\therefore$ The third vertex of the triangle is $(-4, -15)$.

Example 23. If $x - 2y + k = 0$ is a median of the triangle whose vertices are A(-1, 3), B(0, 4) and C(-5, 2), then find the value of k .

Solution. The coordinates of the centroid, say G, of the triangle ABC is

$$\left(\frac{(-1)+0+(-5)}{3}, \frac{3+4+2}{3} \right) \text{ i.e. } (-2, 3).$$

Since centroid G(-2, 3) lies on the median $x - 2y + k = 0$, we have

$$-2 - 2 \times 3 + k = 0 \Rightarrow k = 8.$$

Hence, the value of k is 8.

Example 24. ABC is a triangle and G(4, 3) is the centroid of the triangle. If A, B and C are the points $(1, 3)$, $(4, b)$ and $C(a, 1)$ respectively, find the values of a and b . Also find the length of side BC.

Solution. Since G(4, 3) is the centroid of $\triangle ABC$, we have

$$\frac{1+4+a}{3} = 4 \text{ and } \frac{3+b+1}{3} = 3$$

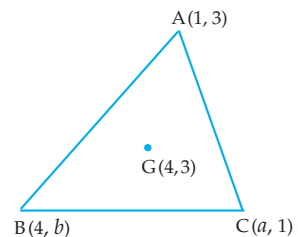
$$\Rightarrow 5 + a = 12 \text{ and } 4 + b = 9$$

$$\Rightarrow a = 7 \text{ and } b = 5.$$

Hence, the values of a and b are $a = 7$, $b = 5$.

The coordinates of the point B and C are $(4, 5)$ and $(7, 1)$ respectively.

$$\begin{aligned} \text{Length of side BC} &= \sqrt{(7-4)^2 + (1-5)^2} \text{ units} \\ &= \sqrt{9+16} \text{ units} = 5 \text{ units.} \end{aligned}$$



Exercise 11

1. Find the coordinates of the mid-points of the line segments joining the following pairs of points :
 (i) $(2, -3), (-6, 7)$ (ii) $(5, -11), (4, 3)$ (iii) $(a + 3, 5b), (2a - 1, 3b + 4)$.
2. The coordinates of two points A and B are $(-3, 3)$ and $(12, -7)$ respectively. P is a point on the line segment AB such that $AP : PB = 2 : 3$. Find the coordinates of P.
3. P divides the distance between A $(-2, 1)$ and B $(1, 4)$ in the ratio 2:1. Calculate the coordinates of the point P.
4. (i) Find the coordinates of the points of trisection of the line segment joining the points $(3, -3)$ and $(6, 9)$.
 (ii) The line segment joining the points $(3, -4)$ and $(1, 2)$ is trisected at the points P and Q. If the coordinates of P and Q are $(p, -2)$ and $\left(\frac{5}{3}, q\right)$ respectively, find the values of p and q .
5. (i) The line segment joining the points A $(3, 2)$ and B $(5, 1)$ is divided at the point P in the ratio 1 : 2 and it lies on the line $3x - 18y + k = 0$. Find the value of k .
 (ii) A point P divides the line segment joining the points A $(3, -5)$ and B $(-4, 8)$ such that $\frac{AP}{PB} = \frac{k}{1}$. If P lies on the line $x + y = 0$, then find the value of k .
6. Find the coordinates of the point which is three-fourth of the way from A $(3, 1)$ to B $(-2, 5)$.
Hint.
 Let P be the required point, then $AP = \frac{3}{4}AB$.
7. Point P $(3, -5)$ is reflected to P' in the x -axis. Also P on reflection in the y -axis is mapped as P''.
 (i) Find the coordinates of P' and P''.
 (ii) Compute the distance P'P''.
 (iii) Find the middle point of the line segment P'P''.
 (iv) On which coordinate axis does the middle point of the line segment PP'' lie?
8. Use graph paper for this question. Take 1 cm = 1 unit on both axes. Plot the points A $(3, 0)$ and B $(0, 4)$.
 (i) Write down the coordinates of A₁, the reflection of A in the y -axis.
 (ii) Write down the coordinates of B₁, the reflection of B in the x -axis.
 (iii) Assign the special name to quadrilateral ABA₁B₁.
 (iv) If C is the mid-point of AB, write down the coordinates of C₁, the reflection of C in the origin.
 (v) Assign the special name to quadrilateral ABC₁B₁.
9. The line segment joining A $(-3, 1)$ and B $(5, -4)$ is a diameter of a circle whose centre is C. Find the coordinates of the point C.
10. The mid-point of the line segment joining the points $(3m, 6)$ and $(-4, 3n)$ is $(1, 2m - 1)$. Find the values of m and n .
11. The coordinates of the mid-point of the line segment PQ are $(1, -2)$. The coordinates of P are $(-3, 2)$. Find the coordinates of Q.
12. AB is a diameter of a circle with centre C $(-2, 5)$. If point A is $(3, -7)$, find
 (i) the length of radius AC.
 (ii) the coordinates of B.

(2013)