

DETERMINANTS

Every square matrix can be associated to a number, which is known as its determinant. If $A = (a_{ij})$ is a square matrix having n number of rows and columns, then determinant of A is denoted by $|A|$, e.g.

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ is a determinant of order 3.}$$

Rows and columns of the determinant:

For the determinant A

I st ;
II nd ;
III rd **Rows**

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

I
II
III **Columns**

Position of a point: a_{ij} represent element in 'i' th row and 'j' th column, e.g.

a_{23} is in 2nd row and 3rd column

a_{32} is in 3rd row and 2nd column.

Order of a determinant: Order 1 $A = |a_{11}|$ (one row and one column)

Order 2 $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$ (Two rows and two columns)

Order 3 $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (Three rows and three columns)

Value of the determinant of order 3 can be obtained by calculating the algebraic sum of products of any one row / column with corresponding co-factors.

Minor of an element: Minor of a_{ij} in matrix A is denoted by M_{ij} and is the determinant of order $(n - 1)$ obtained by leaving the ith row and jth column of A. For example

In matrix $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$; $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and $M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

Co-Factor Of An Element: Co-factor of a_{ij} in matrix A is denoted by C_{ij} and its value is given

by $C_{ij} = (-1)^{i+j} M_{ij}$. For example in $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $C_{11} = (-1)^{1+1} M_{11} = d$ and $C_{21} = (-1)^{2+1} M_{21} = -b$

Important Results : (i) If A is square matrix of order n and $A = k B$ then $|A| = k^n |B|$
(ii) If A is non singular matrix of order n and then $|adj A| = |A|^{n-1}$

Properties of Determinants:

- (i) The value of a determinant remains unchanged if its rows and columns are interchanged.

$$\text{For } A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{and } A' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{then } A = A'.$$

- (ii) If any two rows or columns of a determinant are identical or proportional, value of determinant is 0. Also if all the values of a row/column are zero, value of it is zero.

$$A = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0 \quad \text{Or} \quad A = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 4 & -4 \\ 3 & 6 & 6 \end{vmatrix} = 0; \quad B = \begin{vmatrix} 1 & 0 & 2 \\ -2 & 0 & 3 \\ 5 & 0 & 7 \end{vmatrix} = 0$$

- (iii) If any two rows or columns of a determinant are interchanged, value of determinant is multiplied by -1.

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}$$

- (iv) If each element of a row or column of a determinant is multiplied by a constant k, then value of the determinant gets multiplied by k.

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{and } B = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{then } A = k \cdot B$$

- (v) If to any row or column of a determinant, a multiple of another row or column is added or subtracted, the value of the determinant remains unchanged. (At a time at least one row or column of the determinant should be kept unchanged)

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad R_1 \rightarrow R_1 + kR_2$$

$$\text{OR} = \begin{vmatrix} a_1 - b_1 & b_1 & c_1 \\ a_2 - b_2 & b_2 & c_2 \\ a_3 - b_3 & b_3 & c_3 \end{vmatrix} \quad C_1 \rightarrow C_1 - C_2$$

- (vi) If each element of a row or column of a determinant is expressed as sum of two elements or more terms, then the determinant can be expressed as sum of two determinants keeping all other rows or columns same.

$$A = \begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

- (vii) If A and B are two square matrices then $|AB| = |A| \cdot |B|$

Applications of Determinants

1. Area of triangle: For a triangle having co-ordinates of vertices as $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) .

$$\text{Area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Cor. Points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are collinear if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.

2. Important results: 1. Square matrix A is singular then $|A| = 0$.

2. If A is a square matrix of order n, then $|kA| = k^n |A|$

3. If A is a non-singular matrix of order n, then $|Adj A| = |A|^{n-1}$

4. If A is a square matrix, then $|A^{-1}| = \frac{1}{|A|}$