

DIFFERENTIAL EQUATION

DIFFERENTIAL EQUATION: An equation involving a function and one or more of its derivatives.

Examples of Differential Equations:

$$(1 - x^2) \frac{dy}{dx} - xy = x^2$$

$$\frac{dy}{dx} = 7^{-y}$$

Order and degree of Differential equation

Order: Highest differential coefficient

Degree: Differential equation when expressed as [Polynomial in the derivatives] is the power of highest order derivative

Example: For given differential equation

$$\frac{4\left(\frac{d^2y}{dx^2}\right)^3}{\frac{d^3y}{dx^3}} + \frac{d^3y}{dx^3} = x^2 - 1,$$

order and degree of the differential equation is given by 3 and 2 respectively.

Degree of differential equation is only defined where it is in polynomial form.

Formation of Differential Equation

Let the y, and x be the dependent and independent variable respectively for equation .

Now, step to form differential equation:

Count the number of independent arbitrary constant, (let if 'n')

Number of arbitrary constants will be equal to the order of differential equation.
(n)

Differentiate the given equation 'n' times to eliminate arbitrary constant.

Above elimination equation will be required differential equation.

Solution of Differential Equations

The solution of differential equation is the Relation between the variables involved which satisfies differential equation.

Types of solutions:

1. General solution:

It contains as many as arbitrary constants as the order of the differential equation.

2. Particular solution

The solution obtained by giving particular values to the arbitrary constants in the general solution of differential equation.

Methods for solving the differential equation:

Variable separable

Homogeneous

Linear differential equation

Variable separable Method

If an equation can be written such that variables are separated for integration then the equation can be solved.

EXAMPLE 1 : Solve the differential equation : $x(x^2 - 1) \frac{dy}{dx} = 1$

ANS: $dy = \frac{dx}{x(x^2-1)}$

Put $x^2 - 1 = t$

$$dy = \frac{dt}{2t(t+1)}$$

$$\int dy = \int \frac{dt}{2t(t+1)}$$

$$y = \frac{1}{2} \{ \log |t| - \log |t+1| \} + c$$

EXAMPLE2 : Solve the following differential equation:

$$\sqrt{1+x^2+y^2+x^2y^2}+xy\frac{dy}{dx}=0$$

ANS: The above equation can be written as

$$\int \frac{y}{\sqrt{1+y^2}} dy = - \int \frac{\sqrt{1+x^2}}{x} dx$$

On LHS put $1+y^2 = t$

And in RHS substitute $x = \tan\theta$ we get $\int \frac{\sin\theta}{\sin^2\theta \cos^2\theta} d\theta$

Put $\cos\theta = z$

$$\text{We get RHS} = \frac{-1}{2} \text{Log} \left| \frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1} \right| + \sqrt{1+x^2} + c_1$$

So the answer is

$$\sqrt{1+y^2} = \frac{1}{2} \text{Log} \left| \frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1} \right| - \sqrt{1+x^2} + C$$

Homogeneous Differential Equation

A function $f(x, y)$ is said to be homogeneous of degree n , if

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

Suppose a differential equation can be expressed in the form

$$dy/dx = f(x, y)/g(x, y) = F(y/x)$$

where, $f(x, y)$ and $g(x, y)$ are homogeneous function of same degree. To solve such types of equations, we put $y = vx$

$$\Rightarrow dy/dx = v + x dv/dx.$$

The given equation, reduces to

$$v + x dv/dx = F(v)$$

$$\Rightarrow x dv/dx = F(v) - v$$

$$\therefore dv / F(v) - v = dx / x$$

Hence, the variables are separated in terms of v and x .

EXAMPLE 1: Solve the following initial value problem:

$$x \frac{dy}{dx} \sin\left(\frac{y}{x}\right) + x - y \sin\left(\frac{y}{x}\right) = 0, y(1) = \frac{\pi}{2}$$

$$\frac{dy}{dx} = \operatorname{cosec} \frac{y}{x} - \frac{y}{x} = 0$$

Put $y = v x$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

On substituting

$$\text{We get } \int -\sin v \, dv = \int \frac{dx}{x}$$

$$\cos \frac{y}{x} = \log |x| + c$$

On putting the values we get $c = 0$

$$\text{So } \cos \frac{y}{x} = \log |x|$$

EXAMPLE 2: Show that the differential equation $2ye^{\frac{x}{y}} dx + (y - 2xe^{\frac{x}{y}}) dy = 0$ is homogeneous and find its particular solution, given that $x = 0$, $y = 1$.

$$\text{ANS: } \frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}}$$

Put $x = v y$

$$\frac{dx}{dy} = v + y \frac{dv}{dy}$$

On putting the values and after integrating we have

$$2e^{x/y} + \log |y| = C$$

As

$$x = 0, y = 1, \text{ so } c = 2$$

$$2e^{x/y} + \log |y| = 2 \quad \text{..(1)}$$

Linear Differential Equation

A linear differential equation of the first order can be either of the following forms

(i) $dy / dx + Py = Q$, where P and Q are functions of x or constants.

(ii) $dx / dy + Rx = S$, where R and S are functions of y or constants.

Consider the differential Eq. (i) i.e., $dy / dx + Py = Q$

$$\therefore \quad \text{IF} = e^{\int P dx}$$

Multiply both the sides by $e^{\int P dx}$, we get

$$e^{\int P dx} \frac{dy}{dx} + Pye^{\int P dx} = Qe^{\int P dx}$$

$$\Rightarrow \quad \frac{d}{dx} \left(ye^{\int P dx} \right) = Qe^{\int P dx}$$

$$\text{Integrating, we get} \quad ye^{\int P dx} = \int Qe^{\int P dx} dx + C$$

$$\text{i.e.,} \quad y (\text{IF}) = \int Q (\text{IF}) dx + C$$

$$\text{where, IF} = \text{Integrating Factor} = e^{\int P dx}$$

Similarly, for the second differential equation $dx / dy + Rx = S$, the integrating factor,

$$\text{IF} = e^{\int R dy} \text{ and the general solution is}$$

$$x (\text{IF}) = \int S (\text{IF}) dy + C$$

EXAMPLE: Solve the differential Equation :

$$(1 - x^2) \frac{dy}{dx} - xy = x^2, \quad y(0) = 2$$

ANS: The given differential equation is linear equation

$$\text{I.F.} = \sqrt{1 - x^2}$$

Solution is

$$y \sqrt{1 - x^2} = \int \frac{x^2}{1 - x^2} \sqrt{1 - x^2} dx$$

$$y \sqrt{1 - x^2} = \int \frac{x^2}{\sqrt{1 - x^2}} dx$$

$$= - \int \sqrt{1 - x^2} dx + \int \frac{1}{\sqrt{1 - x^2}} dx + c$$

$$= \frac{-x\sqrt{1-x^2}}{2} + \frac{1}{2} \sin^{-1} x + c$$

$$\text{As } y(0) = 2 \text{ so } c = 2$$

$$y \sqrt{1 - x^2} = \frac{-x\sqrt{1-x^2}}{2} + \frac{1}{2} \sin^{-1} x + 2$$

