

THREE-DIMENSIONAL GEOMETRY

Important Points

1. Point on X-axis is $(x, 0, 0)$; Y axis is $(0, y, 0)$ and on Z axis is $(0, 0, z)$
2. Point on XY-plane is $(x, y, 0)$; YZ plane is $(0, y, z)$ and on XZ plane is $(x, 0, z)$.
3. Point in the space is $P(x, y, z)$.
4. Distance of $P(x, y, z)$ from XY-plane is z ; from YZ plane is x and from XZ plane is y .
5. Distance of $P(x, y, z)$ from X-axis is $\sqrt{y^2+z^2}$; Y axis is $\sqrt{x^2+z^2}$ and Z axis is $\sqrt{x^2+y^2}$.
6. **Distance Formula** For two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Angles of Inclination: For a line inclined at angle of α, β & γ to coordinate axes then these angles are called the angles on inclination.

Direction cosines: For line having angles of inclinations α, β & γ , $l = \cos \alpha$, $m = \cos \beta$, $n = \cos \gamma$ are its direction cosines and $l^2 + m^2 + n^2 = 1$

Direction ratios: $a = rl$, $b = rm$ and $c = rn$ are taken as direction ratios. Direction ratios are denoted by a, b and c and direction ratios of line passing through points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

For a vector $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ a, b, c are direction ratios.

For unit vector $\vec{r} = l\hat{i} + m\hat{j} + n\hat{k}$ l, m, n are direction cosines.

Direction cosines of X-axis are 1, 0, 0. Direction cosines of Y-axis are 0, 1, 0.

Direction cosines of Z-axis are 0, 0, 1.

Converting D. Ratios into D. Cosines: For direction ratios a, b and c , direction cosines can be

$$\text{calculated as } l = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}} \text{ and } n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

STRAIGHT LINE

1. Line passing through point A (P.V. $\vec{a}$) and parallel to $\vec{m}$ is $\vec{r} = \vec{a} + \lambda \vec{m}$

Cartesian form: A (x_1, y_1, z_1) & a, b, c be D. R. of vector parallel to line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

2. A general point on the line is

$$P(a\lambda + x_1, b\lambda + y_1, c\lambda + z_1)$$

3. Line passing through two points A (P.V. $\vec{a}$) & B (P.V. $\vec{b}$) is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

Cartesian form: For points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

4. For lines having angle θ between them

$$\vec{r}_1 = a_1 + \lambda m_1, \vec{r}_2 = a_2 + \mu m_2 \text{ OR } \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} ; \frac{x-x_1}{a_2} = \frac{y-y_1}{b_2} = \frac{z-z_1}{c_2}$$

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{\vec{m}_1 \cdot \vec{m}_2}{|\vec{m}_1| |\vec{m}_2|}$$

For **perpendicular** lines $\vec{m}_1 \cdot \vec{m}_2 = a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

For **parallel** lines $\vec{m}_1 = \mu \vec{m}_2$; $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

5. **Skew lines** in space are lines which are neither parallel nor intersecting.

6. **Shortest distance between two skew lines.**

For two lines $\vec{r}_1 = \vec{a}_1 + \lambda \vec{m}_1$; $\vec{r}_2 = \vec{a}_2 + \mu \vec{m}_2$

$$\text{S. D.} = \frac{[(\vec{a}_2 - \vec{a}_1) \cdot (\vec{m}_1 \times \vec{m}_2)]}{|\vec{m}_1 \times \vec{m}_2|} \quad \text{OR} \quad \text{S.D.} = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\sum (b_1 c_2 - b_2 c_1)^2}}$$

Condition for two lines to intersect $[(\vec{a}_2 - \vec{a}_1) \cdot (\vec{m}_1 \times \vec{m}_2)] = 0$

For parallel lines $\vec{r}_1 = \vec{a}_1 + \lambda \vec{m}$; $\vec{r}_2 = \vec{a}_2 + \mu \vec{m}$ S. D. = $\frac{(\vec{a}_2 - \vec{a}_1) \times \vec{m}}{|\vec{m}|}$

PLANE

1. **General equation** of plane is $a x + b y + c z + d = 0$ Here a, b, c is the d. r. of normal (perpendicular) to plane.

Vector Form $\vec{r} \cdot \vec{n} + d = 0$, here $\vec{n}$ is normal (perpendicular) to plane.

2. **Plane through one point** A (x_1, y_1, z_1) with p.v. $\vec{a}$ and normal to vector $a\hat{i} + b\hat{j} + c\hat{k}$
(Direction ratios a, b, c) $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$; $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ OR $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

3. **Equation of plane through three points** A (x_1, y_1, z_1); B (x_2, y_2, z_2); C (x_3, y_3, z_3)

$$\begin{vmatrix} x - x_1 & x_2 - x_1 & x_3 - x_1 \\ y - y_1 & y_2 - y_1 & y_3 - y_1 \\ z - z_1 & z_2 - z_1 & z_3 - z_1 \end{vmatrix} = 0 \quad \text{OR} \quad \begin{vmatrix} \vec{r} - \vec{a} & \vec{b} - \vec{a} & \vec{c} - \vec{a} \end{vmatrix} = 0$$

4. **Equation of plane having intercepts a, b, c on the axes** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

5. Plane perpendicular to unit vector $\hat{n}$ and at a distance of p from origin $\boxed{\vec{r} \cdot \hat{n} = p}$

Cartesian Form: $l x + m y + n z = p$, here l, m, n are d. cosines of normal to plane.

6. **Equation of plane through a point** A (x_1, y_1, z_1) and

(a) perpendicular to planes $a_1 x + b_1 y + c_1 z + d_1 = 0$; $a_2 x + b_2 y + c_2 z + d_2 = 0$ **or**

(b) parallel to two vectors with direction ratios $a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}$ $a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}$ **or**

(c) parallel to two lines having direction ratios a_1, b_1, c_1 & a_2, b_2, c_2

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

7. **Equation of plane through two points** $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and

(a) perpendicular to plane $a_1x + b_1y + c_1z + d_1 = 0$ or

(b) parallel to one vector $\hat{a} + \hat{b} + \hat{c}$ or

(c) parallel to line having D.R.'s a, b, c

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix} = 0$$

8. **Equation of plane that contains the lines** $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ &

$$\frac{x-x_1}{a_2} = \frac{y-y_1}{b_2} = \frac{z-z_1}{c_2} \quad \text{is} \quad \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ & $\frac{x-x_1}{a_2} = \frac{y-y_1}{b_2} = \frac{z-z_1}{c_2}$ are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

9. **Equation of plane that contains parallel lines** $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ &

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \quad \text{is} \quad \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix} = 0$$

10. **Equation of plane through the line of intersection of planes**

$$\pi_1: a_1x + b_1y + c_1z + d_1 = 0 \text{ \& } \pi_2: a_2x + b_2y + c_2z + d_2 = 0 \text{ is } \pi_1 + \lambda\pi_2 = 0$$

11. **Distance of a point from a plane:**

From point $A(p.v. \vec{a})$

and plane $\vec{r} \cdot \vec{n} + d = 0$

$$p = \frac{|\vec{a} \cdot \vec{n} + d|}{|\vec{n}|}$$

From point $A(x_1, y_1, z_1)$

and the plane $ax + by + cz + d = 0$

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

12. **Angle between the planes:** If θ be the angle between the planes

$$r \cdot \vec{n}_1 + d_1 = 0 \text{ \& } r \cdot \vec{n}_2 + d_2 = 0 \text{ OR } a_1x + b_1y + c_1z + d_1 = 0 \text{ \& } a_2x + b_2y + c_2z + d_2 = 0$$

$$\cos \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

For **perpendicular** planes $\vec{n}_1 \cdot \vec{n}_2 = a_1a_2 + b_1b_2 + c_1c_2 = 0$;

For **parallel** planes $\vec{n}_1 = \mu \vec{n}_2$; $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

13. **Angle between line and plane:** For line $\vec{r} = \vec{a} + \lambda \vec{m}$ & plane $\vec{r} \cdot \vec{n} + d = 0$

OR For line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ & Plane $a_2x + b_2y + c_2z + d_2 = 0$

$$\sin \theta = \frac{|\vec{n} \cdot \vec{m}|}{|\vec{n}| |\vec{m}|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

For line to be parallel to plane $\vec{n} \cdot \vec{m} = a_1a_2 + b_1b_2 + c_1c_2 = 0$

For line to be perpendicular to plane $\vec{n} = \mu \vec{m}$; $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

14. **Equation of line perpendicular** to plane $a x + b y + c z + d = 0$ and passing through the point

$$(x_1, y_1, z_1) \text{ is } \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$