

Linear Programming

Linear programming (LP) or Linear Optimisation may be defined as the problem of maximizing or minimizing a linear function which is subjected to linear constraints. The constraints may be equalities or inequalities. The optimization problems involve the calculation of profit and loss. Linear programming problems are an important class of optimization problems, that helps to find the feasible region and optimize the solution in order to have the highest or lowest value of the function.

Linear programming, mathematical modeling technique in which a linear function is maximized or minimized when subjected to various constraints. This technique has been useful for guiding quantitative decisions in business planning, in industrial engineering.

Linear Programming Problem (LPP)

The linear programming problem in general calls for optimizing a linear function of variables called the objective function subject to a set of linear equations and/or linear inequations called the constraints or restrictions.

Objective Function

The function which is to be optimized (maximized/minimized) is called an objective function.

Constraints

The system of linear inequations (or equations) under which the objective function is to be optimized is called constraints.

Non-negative Restrictions

All the variables considered for making decisions assume non-negative values.

Important Definitions and Results

(i) **Solution of a LPP** A set of values of the variables $x_1, x_2, \dots, x_n$ satisfying the constraints of a LPP is called a solution of the LPP.

(ii) **Feasible Solution of a LPP** A set of values of the variables $x_1, x_2, \dots, x_n$ satisfying the constraints and non-negative restrictions of a LPP is called a feasible solution of the LPP.

(iii) **Optimal Solution of a LPP** A feasible solution of a LPP is said to, be optimal (or optimum), if it also optimizes the objective function of the problem.

(iv) **Graphical Solution of a LPP** The solution of a LPP obtained by graphical method i.e., by drawing the graphs corresponding to the constraints and the non-negative restrictions is called the graphical solution of a LPP.

(v) **Unbounded Solution** If the value of the objective function can be increased or decreased indefinitely, such solutions are called unbounded solutions.

(vi) **Fundamental Extreme Point Theorem** An optimum solution of a LPP, if it exists, occurs at one of the extreme points (i.e., corner points) of the convex Polygon of the set of all feasible solutions.

- **A few important linear programming problems are:**

- (i) Diet problems
- (ii) Manufacturing problems
- (iii) Transportation problems

Theorem 1 Let R be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 2 Let R be the feasible region for a linear programming problem, and let be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R .

- If the feasible region is unbounded, then a maximum or a minimum may not exist. However, if it exists, it must occur at a corner point of R .

Corner point method to solve LPP graphically :

For solving a linear programming problem. The method comprises of the following steps:

- (i) Find the feasible region of the linear programming problem and determine its corner points (vertices).
- (ii) Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m respectively be the largest and smallest values at these points.

(iii) If the feasible region is bounded, M and m respectively are the maximum and minimum values of the objective function.

- If the feasible region is unbounded, then,

(i) M is the maximum value of the objective function, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, the objective function has no maximum value.

(ii) m is the minimum value of the objective function, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, the objective function has no minimum value.

- If two corner points of the feasible region are both optimal solutions of the same type, i.e., both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.

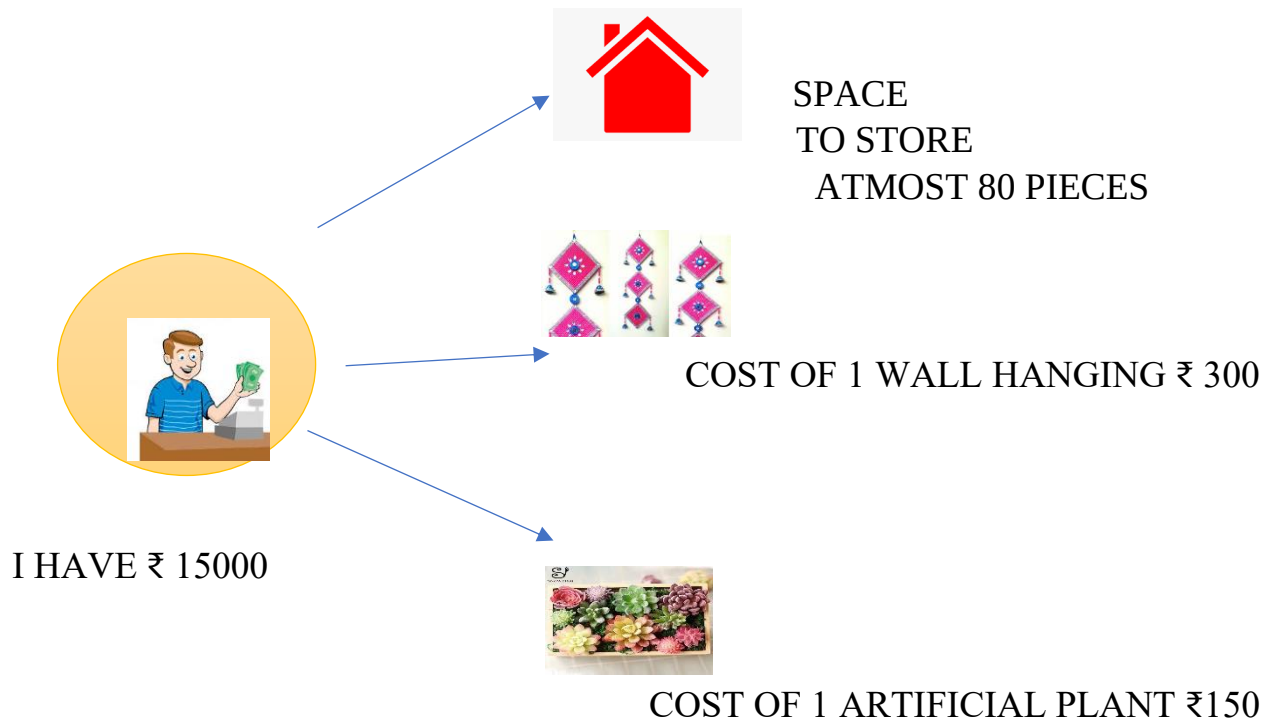
Example:

Suppose a shopkeeper deals in two items-wall hangings and artificial plants.



He has ₹15000 to invest and a space to store almost 80 pieces. A wall hanging costs him ₹300 and an artificial plant ₹150. He can sell a wall hanging at a profit of ₹50 and an artificial plant at a profit of ₹18.

If we assume that he can sell all the items that he buys then try to formulate the given information mathematically.



Let us assume x be the number of wall hangings and y be the number of artificial plants that the dealer sells.

As a wall hanging cost ₹300 and an artificial plant costs ₹150, the cost of x wall hangings and y artificial plants is $300x + 150y$. we are given the dealer can invest at most ₹15000.

hence, we get $300x + 150y \leq 15000$
i.e. $2x + y \leq 100$

As the dealer has space to store at most 80 pieces, we get
 $x + y \leq 80$

As the number of wall hangings and artificial plants can't be negative so we get
 $x \geq 0, y \geq 0$

The dealer wants to invest in such a way so as to maximize his profit, say, Z which stated as a function of x and y is given by

$$Z = 50x + 18y$$

Suppose he sell 30 wall hangings and 30 artificial flowers then the profit will be ₹ 2040 or if he sells 20 walls hanging and 40 artificial flowers then the profit will be ₹1720, in this

We, thus, find that the dealer can invest his money in different ways and he would earn different profits by following different investment strategies. Now we will look for certain strategies so that shopkeeper will get maximum profit. Now first of all let's learn certain terminologies used in LPP.

Let us define some technologies used in Linear programming using the above example.

- **Decision Variables:** The decision variables are the variables that will decide our output. They represent our ultimate solution. To solve any problem, we first need to identify the decision variables. For the above example, x denotes the number of wall hangings and y denotes the number of artificial plants that the dealer buys and sells, are our decision variables.
- **Objective Function:** It is defined as the objective of making decisions. In the above example, the shopkeeper wants max profit represented by Z . So, profit is my objective function
- **Constraints:** The constraints are the restrictions or limitations on the decision variables. They usually limit the value of the decision variables. like $2x + y \leq 100$ (investment constraint) $x + y \leq 80$ (storage constraint) in our example.
- Now, we formulate LPP for the above situation

So, the required LPP is

$$Z = 50x + 18y \text{ (objective function)}$$

➤

$$2x + y \leq 100 \text{ (investment constraint)}$$

$$x + y \leq 80 \text{ (storage constraint)}$$

➤

$$x \geq 0, y \geq 0 \text{ (Non-negative constraints)}$$

Draw lines

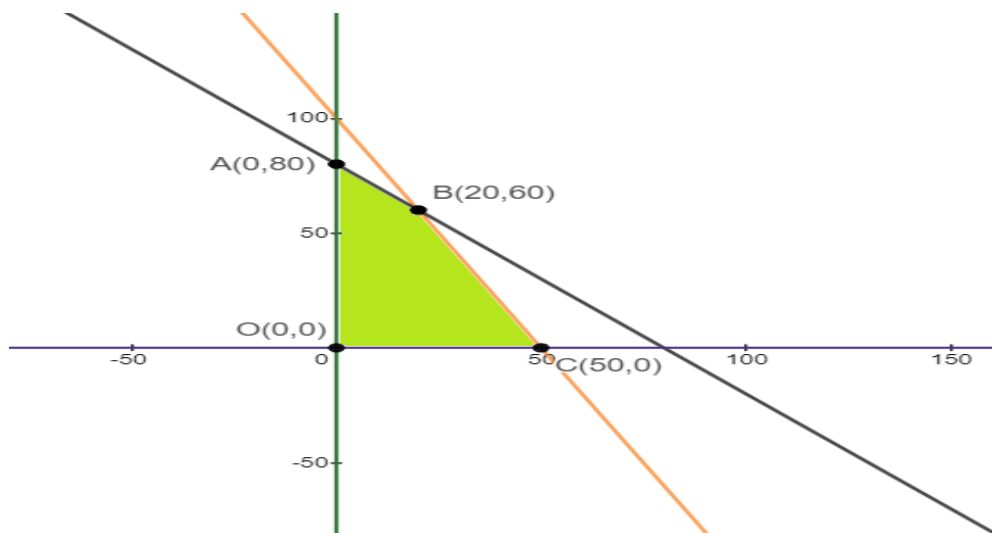
$$2x + y = 100$$

x	0	50
y	100	0

$$x + y = 80$$

x	0	80
y	80	0

Graphical represent inequalities and shade the region satisfied by the given inequations. The shaded region in the adjoining figure shows the feasible region determined by the given constraints. We observe that the feasible region **OABC** is bounded. Points within and on the boundary of the feasible region represent feasible solutions of the constraints. Every point within and on the boundary of the feasible region OABC represents feasible solution to the problem. For example, the point (20,60) is a feasible solution of the problem and so are the points (0, 80), (50, 0) etc. Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an **optimal solution**.



Now we can use CORNER POINT METHOD and get (20,60) as the optimal solution which means 20 are the number of wall hangings and 60 be the number of artificial plants that the dealer sells to have maximum profit.