

Permutations & **combinations**

You walk in a stationary shop.....

1-There are 5 different pencils and 4 different erasers in the store. How many ways are there are to buy a pencil and an eraser?

Solution

First let us choose a pencil from the set of pencils, we have 5 choices. Then choose an eraser from the set of erasers, we have 4 choices. Therefore the number of al possible set of 1 pencil and 1 eraser is 20 ($5 \times 4 = 20$).

2-Also there are 7 different sharpeners. How many ways are there to buy 1 pencil, 1 eraser, 1 sharpener?

Solution

We have 20 possible sets of 1 pencil and 1 eraser. Now, for choosing a sharpener there are 7 choices. Therefore a total of 140 ($20 \times 7 = 140$) possible sets.

In the previous 2 problems we have used the 'The Rule of Product'. The more formal definition is as follows:

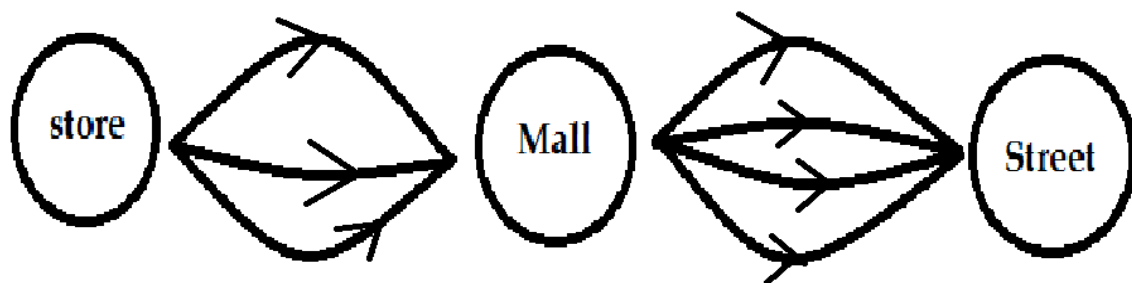
"If an event can occur in m different ways following which another event can occur in n different ways, then the total occurrences of the events in the given order is $m \times n$."

When you walk out of the store.....

3-There are 3 doors to exit the stationary store and 4 doors to exit the mall. In how many ways can you move out of the mall?

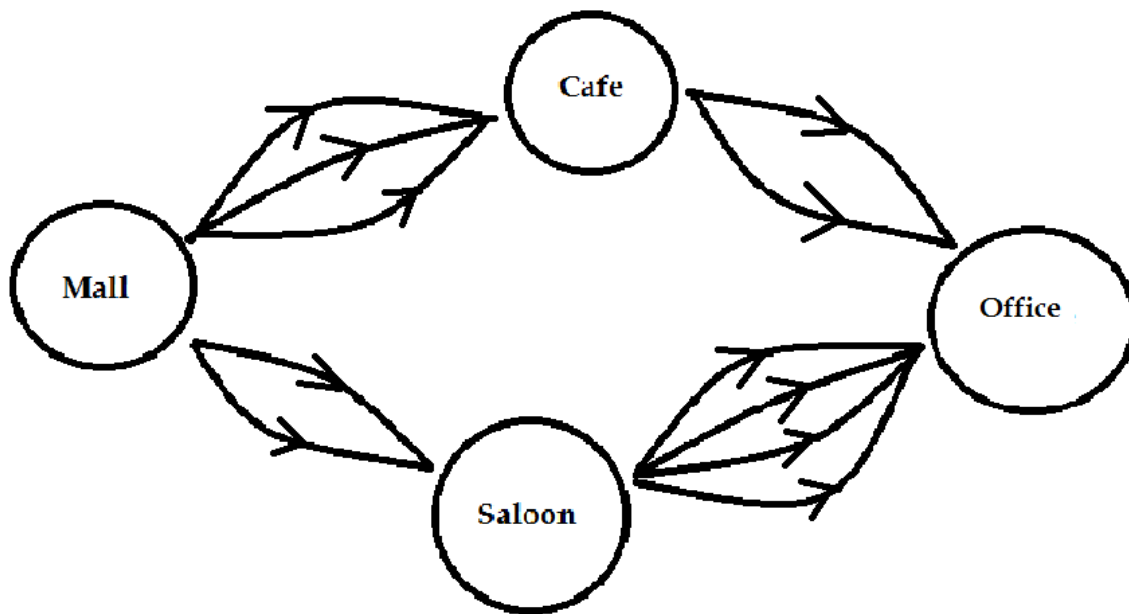
Solution

There are 3 choices to move out from the store and 4 choices of going out of the mall. Hence a total of 12 ways ($3 \times 4 = 12$) to move out.



Route to Office.....

4-There are two routes for you to go to office from the mall as shown in the figure.



In how many ways can you reach the office?

Solution

You can reach the office in 2 ways via café or via saloon.

Via Café

Mall to café; 3 ways .

Café to office; 2 ways.

Total 6 ways ($3 \times 2 = 6$)

Via Saloon

Mall to saloon; 2 ways.

Saloon to office; 4 ways.

Total 8 ways ($2 \times 4 = 8$)

Hence the number of ways for you to reach home can be calculated by adding the numbers, $6 + 8 = 14$.

This problem highlights ‘The Rule of Sum’. The more formal statement is written as follows:

“If an event can occur in m different ways in one way and in n different ways in the other way, then the event can occur in a total of $m + n$ different ways.”

Note: - *The reader is should understand the difference between the rules. The following examples may help.*

Some Examples

1-A room has 6 entrances, in how many ways are there to enter and then leave the room?

[Ans- 6×6]

2-Find the number of ways to colour the unit square of a 2 by 2 table with either black or white colour.

[Ans- 2^4]

3-How many ways are there to choose 1 consonant and 2 vowels from a set of English alphabets?

[Ans- $21 \times 5 \times 4$]

4-A boy has 3 red and 4 blue shirts. How many outfits are there for him today?

[Ans- $3 + 4$]

5-There are 3 rabbits, 2 birds and 6 fishes. If you can choose only one animal as pet. How many choices do you have?

[Ans- $3 + 2 + 6$]

You went to café.....

5-Let there be 5 people (including yourself) outside the café who form a queue. In how many ways can this queue be form?

Solution

There are 5 choices for the first place, after that 4 choices for the second place. Similarly 3 choices for third and 2 choices for

the fourth place and only 1 choice left for the last place, hence the answer is $5 \times 4 \times 3 \times 2 \times 1 = 120$.

Now let us calculate the number of ways to arrange n objects (or people) in a row. Such arrangements are called 'permutations'.

The value of such arrangements is

$n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1 = n!$ (pronounced n factorial).

The following examples may help you deal with this function.

Find the value of

1- $11!/10!$ [Ans- 11]

2- $5!/(5 - 3)!$ [Ans-60]

The above expression can also be written as $5P3 = 5!/(5 - 3)!$ (pronounced 5 permute 3). In more general way as $nPr = n!/(n - r)!$.

3- $6P3 - 5P2$ [Ans-100]

Counting the words.....

How many words with or without meaning can be formed from the word:-

1-"NICE"

Since all 4 letters are different this problem is same as that of arranging the objects in a row.

Hence, the answer is $4! = 24$.

2-"SKY"

Since all 3 letters are different , hence $3!$ words can be made from it.

3-"HELLO"

First let us consider both 'L' as different, therefore the answer becomes $5!$ words.

But since both 'L' can also exchange places within themselves, therefore we are counting every letter twice and hence the answer becomes $\frac{5!}{2!} = 60$ words.

4-"CALL"

There are 5 letters out of which 2 are similar. Therefore

there can be $\frac{5!}{2!}$ words from it.

5-“CARAVAN”

Let us treat 3 As as different namely A_1, A_2, A_3 so we get $8!$ different words. However any word which is formed can be obtained by just interchanging the As is identical.

Therefore we are counting each word $3!$ times more.

Therefore the answer is $\frac{8!}{3!}$.

6-“INTELLIGENT”

We have 2 Is, 2 Ls, 2 Ts, 2 Ns similar. Temporarily thinking all letters as different we get $11!$ words, then

remembering the letter I is different we get $\frac{11!}{2!}$ now

remembering the L is different we get $\frac{11!}{2! \times 2!}$. Using similar

steps we come to $\frac{11!}{2!^4}$.

7-“MATHEMATICAL”

[Ans- $\frac{12!}{2! \times 2! \times 3!}$]

The reader is advised to go through the following exercise thoroughly.

Exercise

1-There are 4 identical apples and 5 identical bananas. In how many ways can you arrange all of the fruits in a row?

$$[\text{Ans}-\frac{9!}{4! \times 5!}]$$

2-Find the number of even numbers greater than 1 and less than 99, using the techniques of combinatorics.

$$[\text{Ans}-4 + (9 \times 5)]$$

3-Let there be a language of only 3 letters namely A, B & C. A word in this language is an arbitrary sequence of no more than 4 letters. How many words are there in this language?

$$[\text{Ans}-3 + 3^2 + 3^3 + 3^4]$$

4-How many 5 digit binary sequences are possible?

$$[\text{Ans}-2^4]$$

5-There are 10 steps in a flight of stairs. When going down you can jump some steps, perhaps even over all 10. How many ways are there to go down?

$$[\text{Ans}-2^{10}]$$

6-How many diagonals are there in a convex n -gon.

$$[\text{Ans}-\frac{n(n-3)}{2}]$$

7-In how many ways can 5 people sit on a circular table?

$$[\text{Ans}-\frac{5!}{5}]$$

8-In how many ways can you make a necklace with 5 different types of pearls in it?

[Ans- $\frac{5!}{5 \times 2}$]

9-How many ways are there to put 8 rooks on a chess board so that they do not attack each other?

[Ans-8!]

10- Find the number of even numbers less than 10,000.

[Ans- $10^4 - 1$]

11- A number is called “odd looking” if all the digits in it are odd. Find the number of odd looking numbers less than 10,000.

[Ans- $5 + 5^2 + 5^3 + 5^4$]

12- Find the number of 2 digit numbers whose sum is even.

[Ans- 9×5]

In the Office.....

There are 3 divisions A, B and C with 30 members each.

1-You have to choose a manager and a secretary from division A. In how many ways can this be done?

Solution

For manager there are 30 choices and after that there are 29 choices for secretary since manager can't be the secretary. Hence a total of 30×29 ways.

2-In division B you have to choose a team of 2 members for a project. In how many ways can this be done?

Solution

You can choose the team in 30×29 ways. But in contrast to previous problem, no matter who was chosen first the team remains the same. So we are counting each team exactly twice, hence the answer is $\frac{30 \times 29}{2}$.

Assume that you need to do the same with a division of n members. This can be done in $\frac{n(n-1)}{2}$ ways.

$\frac{n(n-1)}{2} = nC2$ (pronounced n choose 2).

3-In division C you have to choose a team of 3 members for a project. In how many ways can this be done?

Solution

We can select the members in $30 \times 29 \times 28$ ways. However, each team was counted several times since the team remains the same no matter who we choose first. Since the number of permutations of the members is $3!$.

Therefore we are counting each team $3!$ times more.

Therefore ${}^{30}C_3$ equals $30 \times 29 \times 28 / 3!$

$$\begin{aligned}\text{Similarly } {}^nC_k &= \frac{n \times (n-1) \times (n-2) \times \dots \times (n-k+1)}{k!} \\ &= \frac{n!}{(n-k)! \times k!} \\ &= {}^nP_k / k!\end{aligned}$$

Some Examples

1-How many triangles can be made connecting the vertices of the octagon?

[Ans- 8C_3]

2-How many ways can you choose 10 cards from a deck of 52 cards?

[Ans- ${}^{52}C_{10}$]

3-How many ways can you choose 2 red cards and 3 black cards from a deck of 52 cards?

[Ans- ${}^{26}C_2 \times {}^{26}C_3$]

4- Prove the following equations both using logic and mathematical technique:-

1- $nCk = nC(n - k)$

2- $(n + 1)Ck = nCk + nC(k - 1)$

5- In how many planes does 5 non-parallel and non-concurrent lines divide the plane?

[Ans- $1 + \frac{5(5-1)}{2}$]
